INTEGRAL TRANSFORMS AND BOUNDARY VALUE PROBLEMS*

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1. Introduction. The linear integral transformations of Laplace, Fourier and Hankel† can be used to solve certain classes of linear boundary value problems in partial differential equations that are not adapted to the method of separating variables and using Fourier expansions. When more than one method applies to a given problem, different methods may give different forms of the solution, and a variety of forms can be helpful in determining properties of the solution.

The particular transformations, if any exist, that will reduce a given boundary value problem to a simpler one depend not only upon the ranges of the independent variables but also upon the differential forms that are involved in the problem. It is the purpose of this paper to illustrate how appropriate integral transforms can be determined for a given problem.

The example used in this illustration was selected not because of its significance in applied mathematics but as a fairly simple one that leads to two different types of transforms, one a finite Hankel transform and the other a new modification of the infinite Fourier sine transform. To best illustrate the technique it seems wise to carry out the solution formally.

The method to be illustrated, of beginning with a general integral transformation and then determining its kernel so that the transformed problem is simpler than the original, has been presented during the past several years in one of the author's courses in partial differential equations. When the process is applied to quite general problems it leads to finite and infinite integral transformations whose kernels are characteristic functions of general Sturm-Liouville problems.‡

2. Steady temperatures in a long shaft. Let the positive x-axis be taken as the axis of a cylindrical shaft semi-infinite in length and let the cylindrical coordinate r denote distance from the axis of the shaft. The unit of length is taken as the radius of the cross section. Heat is generated within the shaft, possibly as a result of internal resistance when the shaft is transmitting a torque, at a steady rate per unit volume that is a prescribed function $F(r, x)$. The temperature of the surface $r=1$, $x>0$, is a prescribed function $G(x)$. We assume that the func-

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† Applications of Hankel transforms are presented in a new book by I. N. Sneddon, Fourier Transforms, McGraw-Hill, New York, 1951. The kernels of Hankel transformations are Bessel functions.

‡ The finite integral transformations of this general type have been introduced in connection with ordinary differential equations in a thesis by P. Burgat, Resolution de problèmes aux limites au moyen de transformations fonctionnelles, Université de Neuchatel, Switzerland, 1950.

tions F and G approach zero as x tends to infinity. Let linear heat transfer take place from the base $x=0$ into surroundings at uniform and constant temperature. Then the steady-state temperatures in the shaft are represented by a function $W(r, x)$ that satisfies the equation of conduction

$$\frac{1}{r}(rW_r)_r + W_{xx} = F(r, x) \qquad (0 \leqq r < 1,\ x > 0), \tag{1}$$

where the subscripts denote partial derivatives, and the boundary conditions

$$W(1, x) = G(x) \qquad (x > 0), \tag{2}$$

$$W_x(r, 0) = hW(r, 0) - B \qquad (0 \leqq r < 1), \tag{3}$$

where h and B denote constants and $h \geqq 0$. Also we can require that

$$\lim_{x \to \infty} W(r, x) = 0. \tag{4}$$

The interpretation of the function $W(r, x)$ as the concentration of a substance diffusing within the shaft is also of some interest in the applications.

When the source function $F(r, x)$ involves both of its variables, and it may be a step function in either one or both variables here, the boundary value problem cannot be adapted to the method of separating variables. Methods of using Green's functions transfer the difficulties to the task of finding those functions.

The presence of the variable r in the coefficients of the derivatives in equation (1) is enough to prevent the effective use of either the Laplace or any ordinary Fourier transformation with respect to r. The Laplace transform of W_{xx} with respect to x involves both the functions $W(r, 0)$ and $W_x(r, 0)$; the linear combination of these functions that is prescribed in condition (3) leaves an unknown function of r in the transformed problem.* Similarly, condition (3) is not adapted to the use of the known Fourier transforms with respect to x.

The boundary value problem (1)–(4) may be solved by representing the known and unknown functions by the same type of series or integral in r or x so as to reduce the problem to one of determining the unknown coefficients. Unless the proper type of representation is selected, however, this method leads to awkward approximations. The method illustrated below determines appropriate types of representation that lead to exact solutions.

3. Transformation with respect to r. Let $w(\lambda, x)$ denote an integral transformation of the unknown function $W(r, x)$ of the type

$$T\{W\} = \int_0^1 W(r, x)p(r)\phi(\lambda, r)dr = w(\lambda, x). \tag{5}$$

* The unknown function can be determined in some cases, with persistence and care; this is illustrated for a fairly simple problem in Sec. 72 of the author's book, Modern Operational Mathematics, 1944.

The kernel $\phi(\lambda, r)$, the weight function $p(r)$ and the nature of the parameter λ are to be determined, if possible, so that the problem corresponding to the boundary value problem (1)–(4) will be one in ordinary differential equations with $w(\lambda, x)$ as the unknown function. The new problem should be fully determined by the conditions prescribed in the original problem and by the transformation itself so that, when the solution $w(\lambda, x)$ is found, the function $W(r, x)$ will be given as the inverse transform $T^{-1}\{w\}$. The inverse of our transformation is to be determined from the conditions on ϕ, p and λ.

The result of formally applying T to both members of our partial differential equation (1) can be written

$$T\left\{\frac{1}{r}(rW_r)_r\right\} + \frac{d^2}{dx^2}w(\lambda, x) = f(\lambda, x), \tag{6}$$

where $f(\lambda, x) = T\{F(r, x)\}$. Our transformation should be such that the first term here reduces to a linear function in $w(\lambda, x)$ involving only λ and prescribed boundary values of $W(r, x)$. Now after integrating twice by parts we find that

$$\begin{aligned} T\left\{\frac{1}{r}(rW_r)_r\right\} &= \int_0^1 \frac{1}{r}(rW_r)_r p(r)\phi(\lambda, r)dr \\ &= \int_0^1 \left[r\left(\frac{p}{r}\phi\right)'\right]' W dr + \left[p\phi W_r - \left(\frac{p}{r}\phi\right)' rW\right]_{r=0}^{r=1}, \end{aligned}$$

where the primes denote ordinary derivatives with respect to r. The last integral is proportional to $w(\lambda, x)$ if ϕ and p are such functions that

$$\left[r\left(\frac{p}{r}\phi\right)'\right]' = -\lambda^2 p\phi, \tag{7}$$

where the factor $-\lambda^2$ has been selected as the most convenient factor involving the parameter. Again, as a matter of convenience, the self-adjoint form of the differential equation (7) is preferred; hence we take

$$p(r) = r.$$

Since W and W_r are continuous functions in the interior of the shaft, the above transformation now reduces to

$$T\left\{\frac{1}{r}(rW_r)_r\right\} = -\lambda^2 w(\lambda, x) + \phi(\lambda, 1)W_r(1, x) - \phi'(\lambda, 1)W(1, x). \tag{8}$$

The function $W_r(1, x)$ is not prescribed in our boundary value problem; thus we shall require that its coefficient $\phi(\lambda, 1)$ in equation (8) be zero. If $\phi(\lambda, r)$ is to be a continuous function of r $(0 \leqq r \leqq 1)$ then, in view of equation (7), ϕ should satisfy the singular Sturm-Liouville system

$$(r\phi')' + \lambda^2 r\phi = 0, \quad \phi(\lambda, 1) = 0, \quad |\phi(\lambda, r)| < M \qquad (0 \leqq r \leqq 1), \tag{9}$$

where the constant M may depend on λ. The differential equation here is Bessel's equation with index zero, and the characteristic functions of this system are

$$\phi(\lambda, r) = J_0(\lambda_n r) \qquad (n = 1, 2, \cdots), \tag{10}$$

where the characteristic numbers λ_n are the positive roots of the equation

$$J_0(\lambda) = 0;$$

that is, the homogeneous system (9) has nonzero solutions satisfying the usual requirements as to continuity only when $\lambda = \lambda_n$, and those solutions are the functions (10) except for constant factors.* The numbers λ_n are listed in mathematical tables.

Our characteristic functions are orthogonal, with weight function r, on the interval $0 < r < 1$, and their norms are $J_1^2(\lambda_n)/2$; that is, if n and i are distinct positive integers then

$$\int_0^1 J_0(\lambda_n r) J_0(\lambda_i r) r dr = 0, \qquad \int_0^1 [J_0(\lambda_n r)]^2 r dr = \frac{1}{2} [J_1(\lambda_n)]^2.$$

Moreover any function of r defined on the interval $0 < r < 1$ that satisfies certain conditions of regularity is represented by its generalized Fourier series of those orthogonal functions. If for each fixed x our function $W(r, x)$ satisfies such conditions, then that expansion can be written

$$W(r, x) = 2 \sum_{n=1}^{\infty} w(\lambda_n, x) \frac{J_0(\lambda_n r)}{J_1^2(\lambda_n)}, \tag{11}$$

where

$$w(\lambda_n, x) = \int_0^1 W(r, x) J_0(\lambda_n r) r dr = T\{W\}. \tag{12}$$

Since equation (11) expresses W in terms of w it represents the inverse of our transformation T. Our transformation T, now defined by equation (12), is a finite Hankel transformation.

The transformation (8) can now be written

$$T\left\{\frac{1}{r}(rW_r)_r\right\} = -\lambda_n^2 w(\lambda_n, x) + \lambda_n J_1(\lambda_n) W(1, x) \tag{13}$$

since $J_0'(y) = -J_1(y)$; thus T reduces the differential form in the variable r to an algebraic form in w and the prescribed boundary function $W(1, x)$. In view of condition (2) our equation (6), the transform of equation (1), can now be written

* For the elementary theory of Bessel functions used here see for instance Chapter 8 of the author's book, Fourier Series and Boundary Value Problems, 1941, and the references cited there.

$$\frac{d^2}{dx^2} w(\lambda_n, x) - \lambda_n^2 w(\lambda_n, x) = f(\lambda_n, x) - \lambda_n J_1(\lambda_n) G(x). \tag{14}$$

With the aid of the basic integration formula

$$T\{1\} = \int_0^1 J_0(\lambda_n r) r dr = \frac{J_1(\lambda_n)}{\lambda_n}$$

the transforms of boundary conditions (3) and (4) become, formally,

$$\frac{d}{dx} w(\lambda_n, x)\Big]_{x=0} - h w(\lambda_n, 0) = -\frac{B}{\lambda_n} J_1(\lambda_n), \qquad \lim_{x \to \infty} w(\lambda_n, x) = 0. \tag{15}$$

The problem (14)–(15) is one in ordinary differential equations with constant coefficients. It can be solved by elementary methods, or by using the Laplace transformation with respect to x. If $p_n(x)$ denotes the right-hand member of equation (14), one form of the solution can be written

$$\lambda_n w(\lambda_n, x) = \frac{B J_1(\lambda_n)}{h + \lambda_n} e^{-\lambda_n x} + \int_0^x p_n(y) \sinh [\lambda_n (x - y)] dy - \frac{h \sinh \lambda_n x + \lambda_n \cosh \lambda_n x}{h + \lambda_n} \int_0^\infty p_n(y) e^{-\lambda_n y} dy.$$

The formal solution of our problem (1)–(4) is obtained by substituting this expression for $w(\lambda_n, x)$ into our inversion formula (11).

4. Transformation with respect to x. We now solve the same boundary value problem (1)–(4) with the aid of an integral transformation of the type

$$S\{W(r, x)\} = \int_0^\infty W(r, x) \theta(\alpha, x) dx = w(r, \alpha). \tag{16}$$

The kernel $\theta(\alpha, x)$ is to be determined so that the problem in the transform $w(r, \alpha)$ is one in ordinary differential equations involving the parameter α and the prescribed boundary values of W. The formal application of S to the members of equation (1) gives the equation

$$\frac{1}{r} \frac{d}{dr} \left\{ r \frac{dw}{dr} \right\} + S\{W_{xx}(r, x)\} = f(r, \alpha), \tag{17}$$

where $f(r, \alpha) = S\{F(r, x)\}$. This is an ordinary differential equation in $w(r, \alpha)$ if $S\{W_{xx}\}$ reduces to an algebraic form in w.

After integrating twice by parts we find that

$$S\{W_{xx}\} = \int_0^\infty \theta W_{xx} dx = \int_0^\infty W \theta'' dx + [\theta W_x - \theta' W]_{x=0}^{x=\infty},$$

where the primes indicate differentiation with respect to x. We now assume

that our prescribed functions $F(r, x)$ and $G(x)$ are such that $W_x(r, x)$, as well as $W(r, x)$, approaches zero as x tends to infinity, an assumption that seems justified in the physical problem. Let θ and θ' be bounded when $x \geqq 0$. According to condition (3), $W_x = hW - B$ when $x = 0$. Thus

$$S\{W_{xx}\} = \int_0^\infty W\theta'' dx + [\theta'(\alpha, 0) - h\theta(\alpha, 0)]W(r, 0) + B\theta(\alpha, 0). \tag{18}$$

Since the function $W(r, 0)$ is not prescribed in our boundary value problem the coefficient in the brackets should vanish. The integral is proportional to w if θ'' is proportional to θ. It is convenient to select $-\alpha^2$ as the factor of proportionality. Then θ satisfies the conditions

$$\theta''(\alpha, x) + \alpha^2\theta(\alpha, x) = 0 \qquad (x > 0), \tag{19}$$

$$\theta'(\alpha, 0) - h\theta(\alpha, 0) = 0, \quad |\theta(\alpha, x)| < M \qquad (x \geqq 0), \tag{20}$$

where M is a constant.

Our problem in the kernel θ is an extension of the Sturm-Liouville type of problem to the semi-infinite interval. The characteristic values of the parameter α consist of all real positive numbers. A convenient form of the solution is

$$\theta(\alpha, x) = \sin [\alpha x + q(\alpha)]$$

where

$$\sin q(\alpha) = \frac{\alpha}{\sqrt{h^2 + \alpha^2}}, \qquad \cos q(\alpha) = \frac{h}{\sqrt{h^2 + \alpha^2}}.$$

Our transformation (16) is thus a modified Fourier transformation

$$S\{W(r, x)\} = \int_0^\infty W(r, x) \sin [\alpha x + q(\alpha)]dx = w(r, \alpha) \tag{21}$$

under which, according to equation (18), the image of W_{xx} is a linear combination of w and of the value of $W_x - hW$ at $x = 0$, namely,

$$S\{W_{xx}(r, x)\} = -\alpha^2 w(r, \alpha) + \frac{\alpha B}{\sqrt{h^2 + \alpha^2}}. \tag{22}$$

The expansion of an arbitrary function $H(x)$, defined on the interval $x \geqq 0$ and satisfying regularity conditions like those used in the ordinary Fourier integral representations, is given by the formula

$$H(x) = \frac{2}{\pi}\int_0^\infty \sin [\alpha x + q(\alpha)]d\alpha \int_0^\infty H(y) \sin [\alpha y + q(\alpha)]dy \quad (x > 0).$$

A method of establishing this extension of the Fourier integral formula is indi-

cated in the literature.* Since the second integral here represents $S\{H(x)\}$ the formula represents the inverse of our transformation S. For each fixed r then

$$W(r, x) = \frac{2}{\pi}\int_0^\infty \sin\,[\alpha x + q(\alpha)]w(r, \alpha)d\alpha. \tag{23}$$

In view of formula (22) the transform (17) of our partial differential equation (1) becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) - \alpha^2 w = f(r, \alpha) - \frac{\alpha B}{\sqrt{h^2+\alpha^2}} \qquad (0 \leqq r < 1), \tag{24}$$

where we have used the boundary conditions (3) and (4). The transform of the remaining condition (2), and the condition that $W(r, x)$ is to be continuous when $0 \leqq r < 1$ can be written

$$w(1, \alpha) = g(\alpha), \qquad |\,w(r, \alpha)\,| < N \qquad (0 \leqq r < 1), \tag{25}$$

where $g(\alpha) = S\{G(x)\}$ and N is some constant. Equation (24) is the nonhomogeneous form of Bessel's equation with index zero. If $p(r, \alpha)$ denotes the right-hand member of equation (24), the solution of problem (24)–(25), obtained by the method of variation of parameters, can be written in the form

$$w(r, \alpha) = g(\alpha)\frac{I_0(\alpha r)}{I_0(\alpha)} - I_0(\alpha r)\int_r^1 \frac{dy}{yI_0^2(\alpha y)}\int_0^y p(t, \alpha)I_0(\alpha t)t\,dt, \tag{26}$$

where $I_0(y) = J_0(iy)$.

The solution of our boundary value problem (1)–(4) is obtained by substituting the expression (26) for $w(r, \alpha)$ into formula (23). This integral form of $W(r, x)$ is substantially different from the series representation of the same function found in Section 3.

Another representation of $W(r, x)$ in terms of infinite series and integrals can be obtained by using the Hankel transformation (12) and its properties (13) and (11) to solve problem (24)–(25).

* Page 210 of the author's book, Modern Operational Mathematics, 1944.

EUCLID'S ALGORITHM AND THE LEAST-REMAINDER ALGORITHM

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1. Introduction. Euclid's algorithm* for finding the greatest common divisor (g.c.d.) of two positive integers $b_1 > b_2$ may be written as follows:

$$
\begin{aligned}
b_1 &= b_2 q_3 + b_3 \\
b_2 &= b_3 q_4 + b_4 \\
&\cdots\cdots\cdots \\
b_{n-1} &= b_n q_{n+1} + b_{n+1} \\
b_n &= b_{n+1} q_{n+2},
\end{aligned}
\tag{E}
$$

where the conditions $b_2 > b_3 \cdots > b_{n+1} > 0$ determine the integers q_k and b_k uniquely. The g.c.d. of b_1 and b_2 is b_{n+1}.

The least-remainder algorithm for finding the g.c.d. of two positive integers $a_1 > a_2$ may be written as follows:

$$
\begin{aligned}
a_1 &= a_2 p_3 + \epsilon_3 a_3 \\
a_2 &= a_3 p_4 + \epsilon_4 a_4 \\
&\cdots\cdots\cdots\cdots \\
a_{m-1} &= a_m p_{m+1} + \epsilon_{m+1} a_{m+1} \\
a_m &= a_{m+1} p_{m+2},
\end{aligned}
\tag{L}
$$

where $\epsilon_k = \pm 1$, $k = 3, 4, \cdots, m+1$, and is determined uniquely by the condition that the positive integers a_k, p_k be such that

$$
0 < a_k < \frac{a_{k-1}}{2}, \qquad k = 3, 4, \cdots, m, \tag{1}
$$

and

$$
0 < a_{m+1} \leqq \frac{a_m}{2}. \tag{2}
$$

If the equality sign occurs in (2) there is a choice for ϵ_{m+1}, and in this case we require that $\epsilon_{m+1} = +1$.

Kronecker has proved that for two given integers the least-remainder algorithm is not longer than any other algorithm for finding the g.c.d. In particular, if $a_1 = b_1$ and $a_2 = b_2$, then $n \geqq m$. If we restrict ourselves to Euclid's algorithm, and the least-remainder algorithm, this qualitative result may be put into a more

* Elementary Number Theory, Uspensky and Heaslet, McGraw-Hill, New York 1939, pp. 26–28 and pp. 45–51. All of the material required for reading this paper may be found in these few pages.

precise form, namely

THEOREM I. *If $a_1=b_1$ and $a_2=b_2$, then the number of equations in set E exceeds the number of equations in set L by just the number of negative signs in the sequence $\epsilon_3, \epsilon_4, \cdots, \epsilon_{m+1}$, that is*

$$n - m = \frac{1}{2}\sum_{k=3}^{m+1}\{|\epsilon_k| - \epsilon_k\}. \tag{3}$$

COROLLARY. *E and L have the same length if and only if they are identical.*

2. Proof of Theorem I. We construct a sequence of algorithms $E\equiv E_0$, $E_1, E_2, \cdots, E_r\equiv L$. In this sequence each E_k will have one less equation than E_{k-1}, and one more negative sign among the remainders $\epsilon_j a_j$. The theorem will then follow immediately from the fact that $E_0\equiv E$ has no negative signs.

Let us examine in E, the set of ratios $R_0=\{r_j\}$ where

$$0 < r_j = \frac{b_j}{b_{j-1}} < 1, \qquad j = 3, 4, \cdots, n. \tag{4}$$

Suppose that $r_j<1/2$ for $j=3, 4, \cdots, k\leqq n-1$, and that $r_{k+1}>1/2$. Then in equation sets E and L the first $k-2$ equations are identical. If no such k exists, then E and L coincide and since there are no negative signs in the sequence $\epsilon_3, \epsilon_4, \cdots, \epsilon_{m+1}$, in this case the theorem is established.

Suppose however that there is a $k\leqq n-1$ as described above. Consider the three equations

$$b_{k-1} = b_k q_{k+1} + b_{k+1}, \qquad 1 > \frac{b_{k+1}}{b_k} > \frac{1}{2}, \tag{5}$$

$$b_k = b_{k+1}q_{k+2} + b_{k+2}, \tag{6}$$

$$b_{k+1} = b_{k+2}q_{k+3} + b_{k+3}. \tag{7}$$

Clearly we have

$$b_{k-1} = b_k(q_{k+1} + 1) - (b_k - b_{k+1})$$

or

$$b_{k-1} = b_k p_{k+1} - a_{k+1}, \qquad 0 < \frac{a_{k+1}}{b_k} < \frac{1}{2}, \tag{8}$$

and this equation will now coincide with the $(k-1)$th equation of the set L. Further since $r_{k+1}>1/2$, $q_{k+2}=1$ and equation (6) must have the form

$$b_k = b_{k+1} + b_{k+2} \tag{9}$$

and hence

$$a_{k+1} = b_k - b_{k+1} = b_{k+2}. \tag{10}$$

Finally from (7) we have

$$b_{k+1} + b_{k+2} = b_{k+2}(q_{k+3} + 1) + b_{k+3}$$

which together with (9) and (10) gives

$$b_k = a_{k+1}(q_{k+3} + 1) + b_{k+3}. \tag{11}$$

The set E_1 is now obtained from E_0 by suppressing the equations (5), (6), and (7) and replacing them by equations (8) and (11). One equation has been lost and one minus sign has been added, namely the one in equation (8), and the first $k-1$ equations in E_1 and L are identical. For the set of equations E_1, there is a new set of ratios of remainders which we denote by R_1, consisting of the ratios

$$\frac{b_3}{b_2}, \frac{b_4}{b_3}, \cdots, \frac{b_k}{b_{k-1}}, \frac{a_{k+1}}{b_k}, \frac{b_{k+3}}{a_{k+1}}, \frac{b_{k+4}}{b_{k+3}}, \cdots, \frac{b_n}{b_{n-1}}. \tag{12}$$

If $k=n-1$ then $b_{k+3}=b_{n+2}=0$ and the sequence (12) has as its last term a_{k+1}/b_k.

The set of ratios R_1 is again examined and if one of these ratios exceeds one-half, we select the first such ratio and proceed to generate the set E_2 from the set E_1, by the same method used in obtaining E_1 from E_0. After a finite number of such steps, we arrive at an equation set E_ν with a set of ratios of remainders R_ν, none of which exceed one-half. Then $E_\nu \equiv L$ and Theorem 1 is proved.

3. Further results. The method developed in the preceding section can be used to prove two other theorems on the relative lengths of E and L.

THEOREM 2. *Let $a_1=b_1$ and $a_2=b_2$. Divide the set of ratios R_0 into blocks consisting only of those ratios which exceed one-half. A block may have only one member but each block is to be as long as possible. Number the ratios in each block consecutively beginning with the number one for each block. Let μ be the number of odd-numbered ratios. Then*

$$n - m = \mu. \tag{13}$$

Example: If $b_1=1065$ and $b_2=674$, the set R_0 is

$$\underline{\underset{1}{\frac{391}{674}}, \underset{2}{\frac{283}{391}}}, \frac{108}{283}, \underline{\underset{1}{\frac{67}{108}}, \underset{2}{\frac{41}{67}}, \underset{3}{\frac{26}{41}}, \underset{4}{\frac{15}{26}}, \underset{5}{\frac{11}{15}}}, \frac{4}{11}, \underline{\underset{1}{\frac{3}{4}}}.$$

The blocks are indicated by heavy lines and the number of each ratio is indicated just above the ratio. In this case $\mu=5$.

Proof. Just as in the preceding sections we construct a sequence of algorithms $E \equiv E_0, E_1, \cdots, E_\nu = L$. The ratio set R_1 for E_1 is given by (12). In going from E_0 to E_1 the ratio b_{k+1}/b_k, which is an odd-numbered ratio, is replaced by $a_{k+1}/b_k < 1/2$, and one equation is lost. The ratio b_{k+2}/b_{k+1} disappears, but either

$b_{k+2}/b_{k+1}<1/2$ or it is an even-numbered ratio. Finally from equation (10) $a_{k+1}=b_{k+2}$ and hence all succeeding ratios are unchanged, and the next ratio exceeding one-half will be odd-numbered. This completes the proof of Theorem 2.

Although a little awkward, Theorem 2 has the merit of being a complement to Theorem 1. For given the set L one can determine by Theorem 1 how many equations will be in E, and conversely given the set E one can determine by Theorem 2, how many equations will be in L.

THEOREM 3. *Let $a_1=b_1$ and $a_2=b_2$. Then in E and L*

$$n+\sum_{j=3}^{n+2} q_j = m+\sum_{j=3}^{m+2} p_j, \tag{14}$$

that is, the number of equations plus the sum of the quotients is constant.

Proof. We only need to observe that in replacing equations (5), (6), and (7) by (8) and (11) the change in the quotient sum is $+1$. For $q_{k+2}=1$ is lost by the suppression of equations (6) and q_{k+1} and q_{k+3} are increased by 1 in equations (8) and (11). Theorem 3 is proved.

Theorem 3, and the corollary to Theorem 1, are no longer valid if other g.c.d. algorithms are considered. In fact Uspensky and Heaslet have already noted that a g.c.d. algorithm may be the same length as the Euclid algorithm for the same two positive integers, without the two algorithms being identical. It would be interesting to see in what way the above results can be modified to include other g.c.d. algorithms. In particular, it would be nice to have some expression such as (14) which would be invariant for all g.c.d. algorithms.

A RECURRENCE RELATION FOR THREE-LINE LATIN RECTANGLES

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1. Introduction. This note gives a derivation of a new recurrence relation for the number of reduced three-line Latin rectangles. It may be recalled from earlier papers [2, 3] that a Latin rectangle is an array in which each row is a permutation of n elements and each column has unlike elements, and that it is reduced when the first row is in standard order.

The relation is "impure," in the sense that it contains an auxiliary function, but it is of lower order than the pure recurrence of Kerawala [1] and easier to compute. The structure of residues modulo p, a prime, is an almost immediate consequence. Finally it serves to simplify the derivation of the elegant asymptotic formula of Yamamoto [4, 5].

2. Derivation of recurrence relation. Write $L(3, n)$ for the number of three-line Latin rectangles and $K(3, n)$, abbreviated to K_n, for the corresponding reduced number ($L(3, n) = n!\, K(3, n)$). Then the basic formula used in the first of Yamamoto's papers, of which I have an English translation through the kindness of the author, may be written as follows.

$$K_n = \sum_0^n \binom{n}{r} 2^r v_{n-r} = (v + 2)^n \tag{1}$$

with the last a symbolic form and

$$v_n = n! \sum_0^n (-1)^s (n - s)! \binom{3n - 2s + 2}{s} = n!\, u_n, \text{ say.} \tag{2}$$

Using the identity

$$\frac{3n}{3n - 2s}\binom{3n - 2s}{s} = \binom{3n - 2s + 2}{s} - 3\binom{3n - 2s}{s - 2} + 2\binom{3n - 2s - 1}{s - 3}$$

multiplied throughout by $(-1)^s(n-s)!$ and summed on s, it follows that

$$u_n = nu_{n-1} + 3u_{n-2} + 2u_{n-3} + 3(-1)^n - 2\delta_{0n} \tag{3}$$

with δ a Kronecker symbol ($\delta_{00}=1$, $\delta_{0n}=0$, $n \neq 0$).

Hence

$$v_n = n^2 v_{n-1} + 3(n)_2 v_{n-2} + 2(n)_3 v_{n-3} + 3n!(-1)^n - 2\delta_{0n}, \tag{4}$$

$(n)_k = n(n-1) \cdots (n-k+1)$ being a factorial symbol.

It is now convenient to use the exponential generating function:

$$\exp vt = \sum_0^\infty v_n t^n/n!$$

with the left-hand side symbolic. By (1)

$$\exp Kt = \sum_0^\infty K_n t^n/n! = \exp (v + 2)t \tag{5}$$

and by (4)

$$[1 - t - t^2(v + 3) - 2t^3] \exp vt = 3(1 + t)^{-1} - 2, \tag{6}$$

so that

$$[1 - t - t^2(K + 1) - 2t^3] \exp Kt = e^{2t}[3(1 + t)^{-1} - 2] = \exp kt, \text{ say,} \tag{7}$$

and

$$K_n = n^2K_{n-1} + (n)_2K_{n-1} + 2(n)_3K_{n-3} + k_n, \tag{8}$$

the new recurrence formula.

Since

$$(1 + t) \exp kt = e^{2t}(1 - 2t)$$

it follows at once that

$$k_n + nk_{n-1} = -(n - 1)2^n \tag{9}$$

which may be used to calculate k_n. The following short table is introduced for numerical concreteness.

n	0	1	2	3	4	5	6	7
k_n	1	−1	−2	−10	−8	−88	208	−2,224
K_n	1	0	0	2	24	552	21,280	1,073,760

Recurrence (9) may be used to eliminate k_n in (8); the further elimination of $(n-1)2^n$ yields Kerawala's recurrence which I do not take space to write.

3. Asymptotic expansion. As $k_n = O(n!)$ it may be ignored for asymptotic results, and substitution of

$$n!^{-2}e^3K_n = 1 + \frac{a_1}{n} + \frac{a_2}{(n)_2} + \cdots + \frac{a_s}{(n)_s} + \cdots \tag{10}$$

into recurrence (8) and use of the identity

$$\frac{1}{(n-1)_s} = \frac{1}{(n)_s} + \frac{s}{(n)_{s+1}}$$

shows that

$$(s + 1)a_{s+1} + a_s + 2a_{s-1} = 0, \tag{11}$$

so that

$$a_s = H_s(-1/2)/s! \tag{12}$$

with H_s an Hermite polynomial, which is Yamamoto's result.

4. Arithmetical structure. By (9) and Fermat's theorem, with p an odd prime,

$$k_p \equiv 2 \ (p). \tag{13}$$

Since $k_0 = 1$ and $(p+n-1)2^{p+n-1} \equiv 2(n-1)2^n$, this and (9) show that

$$k_{n+p} \equiv 2k_n, \tag{14}$$

$$k_{n+xp} \equiv 2^x k_n. \tag{15}$$

Hence the period of these numbers is xp where $2^x \equiv 1\ (p)$; x is a divisor of $p-1$.

The structure of the K_n is exactly the same, because by (8)

$$K_p \equiv k_p \equiv 2 \equiv 2K_0\ (p) \tag{16}$$

and

$$K_{p+n} \equiv n^2 K_{p+n-1} + (n)_2 K_{p+n-2} + 2(n)_3 K_{p+n-3} + k_{p+n}\ (p)\ .$$

which with (16) entails

$$K_{p+n} \equiv 2K_n. \tag{17}$$

It is interesting to notice that $K_{p+2} \equiv K_{p+1} \equiv 0\ (p)$ is concordant with the relation $K_n = (n-1)(n-2)J_n$ where J_n is the number of rectangles with first column as well as first row standardized, and hence an integer. Thus the numbers $K_0 = 1$, $K_1 = K_2 = 0$, which are not defined combinatorially because there are no such rectangles, may be taken as defined by this circumstance, as appears naturally in the development.

A table of residues, which is easy to make for small moduli, then serves to check the tables of K_n (Kerawala [1], $n = 2(1)15$, Yamamoto [5], $n = 2(1)20$) and any extensions, like the one I have made to $n = 25$, but I do not take space for these large numbers (50 digits for $n = 25$).

From such a residue table curiosities like the following are read off at once:

$$K_n \equiv 0\ (19), \qquad n \equiv 6, 18\ (19);$$
$$K_n \equiv 0\ (23), \qquad n \equiv 5, 10, 11\ (23).$$

The complete factorization of K_n, and the arithmetical structure, yield results like

$$K_n \equiv 0\ (2237), \qquad n \equiv 7\ (2237);$$
$$K_n \equiv 0\ (26{,}153), \qquad n \equiv 8\ (26{,}153).$$

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ON THE FINITE DIFFERENCE ANALOGUE OF RODRIGUES' FORMULA*

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1. It is well known that several important systems of orthogonal polynomials can be obtained in the form

$$\frac{1}{\rho(x)}\frac{d^n}{dx^n}\{\rho(x)[X(x)]^n\}; \qquad n = 0, 1, 2, \cdots, \tag{1}$$

where $\rho(x)$ is an infinitely differentiable function and $X(x)$ is a polynomial, both independent of n. One may ask for all functions $\rho(x)$ and polynomials $X(x)$ for which (1) is a polynomial of exact degree n for each n. As far as we know this question was first put, slightly differently, and completely answered, by Tricomi [8]. By a slight modification of Tricomi's argument one sees that the degree of $X(x)$ cannot exceed 2, and that all polynomials generated in the manner of (1) may be reduced to one of the "classical" orthogonal polynomials by a linear change of the independent variable.

There are also some systems of polynomials generated similarly by finite differences in the form:

$$p_n(x) = \frac{1}{\rho(x)}\Delta^n[\rho(x-n)X(x)X(x-1)\cdots X(x-n+1)]; \qquad n = 0, 1, 2, \cdots, \tag{2}$$

where $\rho(x)$ is any function, $X(x)$ is a polynomial, both are independent of n, and Δ is the operator of finite differences defined by

$$\Delta f(x) = f(x+1) - f(x), \qquad \Delta^{n+1}f(x) = \Delta(\Delta^n f(x)), \qquad n = 0, 1, 2, \cdots. \tag{3}$$

One may again ask for all functions $\rho(x)$ and all polynomials $X(x)$ for which (2) is a polynomial of exact degree n for each n. The answer is surprisingly simple. $X(x)$ must be of degree $\leqq 2$, $\rho(x)$ is a combination of gamma functions, and all systems of polynomials capable of a representation in the form (2) can be enumerated.

Actually, W. Hahn [3], has discussed the corresponding problem for the operator $\mathfrak{d}$ defined by

$$\mathfrak{d}f(x) = \frac{f(qx) - f(x)}{(q-1)x}$$

(with a restriction on the degree of $X(x)$). Both d/dx and Δ are limiting cases of $\mathfrak{d}$. Since Hahn's work introduces of necessity basic hypergeometric series, and since the limiting process $\mathfrak{d} \to \Delta$ is quite involved, it seemed worth while to give a simple and elementary discussion of (2).

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2. We proceed to determine all the sets of polynomials (2), each $p_n(x)$ of exact degree n. In the following we exclude tacitly all values of the parameters which make the formulas meaningless. We shall first prove that if the degree k of X exceeds 2, we do not obtain a required set, because for some n the degree of $p_n(x)$ is larger than n.

Let $k \geqq 2$. We write (2) for $n=1$ and get

$$p_1(x) = [\rho(x)]^{-1}\Delta[\rho(x-1)X(x)],$$

which gives

$$\frac{\rho(x-1)}{\rho(x)} = \frac{X(x+1) - p_1(x)}{X(x)}. \tag{4}$$

Factoring the polynomials on the right we have

$$\frac{\rho(x-1)}{\rho(x)} = \prod_{j=1}^{k} \frac{(x+\alpha_j)}{(x+\beta_j)}. \tag{5}$$

A solution of this equation is

$$\rho(x) = \prod_{j=1}^{k} \frac{\Gamma(x+\beta_j+1)}{\Gamma(x+\alpha_j+1)}.$$

Since (2) does not depend on the particular solution of (5) chosen, we have

$$p_n(x) = \prod_{j=1}^{k} \frac{\Gamma(x+\alpha_j+1)}{\Gamma(x+\beta_j+1)}\,\Delta^n\left\{\prod_{l=1}^{k} \frac{\Gamma(x+\beta_l+1)}{\Gamma(x+\alpha_l-n+1)}\right\} \tag{6}$$

and therefore

$$p_1(x) = X(x+1) - \prod_{j=1}^{k}(x+\alpha_j). \tag{7}$$

Combining (6) for $n=2$ with (7) we get

$$p_2(x) = X(x+1)\Delta^2 X + p_1(x)\Delta X + X(x+1)\Delta p_1 + p_1(x)p_1(x-1).$$

If $k>2$ this clearly makes $p_2(x)$ of degree larger than 2. We shall therefore only consider $k \leqq 2$.

Case 1: $k=2$.

From (6) we obtain

$$p_n(x) = \frac{\Gamma(x+\alpha_1+1)\Gamma(x+\alpha_2+1)}{\Gamma(x+\beta_1+1)\Gamma(x+\beta_2+1)}\,\Delta^n\left[\frac{\Gamma(x+\beta_1+1)\Gamma(x+\beta_2+1)}{\Gamma(x+\alpha_1-n+1)\Gamma(x+\alpha_2-n+1)}\right] \tag{8}$$

and

$$p_n(x) = \sum_{p=0}^{n} \binom{n}{p} (-1)^p$$

$$\times \frac{\Gamma(x+\beta_1+1+n-p)\Gamma(x+\beta_2+1+n-p)\Gamma(x+\alpha_1+1)\Gamma(x+\alpha_2+1)}{\Gamma(x+\beta_1+1)\Gamma(x+\beta_2+1)\Gamma(x+\alpha_1+1-p)\Gamma(x+\alpha_2+1-p)}.$$

Using the identity

$$\Gamma(z-p) = (-1)^p \frac{\Gamma(z)\Gamma(1-z)}{\Gamma(p+1-z)}, \qquad p \text{ an integer},$$

we obtain

$$p_n(x) = \frac{\Gamma(x+\beta_1+1+n)\Gamma(x+\beta_2+1+n)}{\Gamma(x+\beta_1+1)\Gamma(x+\beta_2+1)}$$

$$\times \sum_{p=0}^{n} \binom{n}{p} (-1)^p \frac{\Gamma(-x-\beta_1-n)\Gamma(-x-\beta_2-n)}{\Gamma(-x-\beta_1-n+p)\Gamma(-x-\beta_2-n+p)}$$

$$\times \frac{\Gamma(-x-\alpha_1+p)\Gamma(-x-\alpha_2+p)}{\Gamma(-x-\alpha_1)\Gamma(-x-\alpha_2)},$$

which can be written

$$p_n(x) = \frac{\Gamma(x+\beta_1+1+n)\Gamma(x+\beta_2+1+n)}{\Gamma(x+\beta_1+1)\Gamma(x+\beta_2+1)} \tag{9}$$

$$\times {}_3F_2(-n,\ x-\alpha_1,\ -x-\alpha_2;\ -x-\beta_1-n,\ -x-\beta_2-n;\ 1).$$

A particular case of (9) are the polynomials investigated by Tchebichef (cf. Szegö [7])

$$\alpha_1 = \beta_1 = 0; \qquad \alpha_2 = \beta_2 = -N; \qquad \rho(x) = 1.$$

In order to prove that (9) defines polynomials of degree n and that these are identical with a class of polynomials mentioned by Hahn, we use a transformation formula to be established in section 3. According to (18)

$$p_n(x) = \frac{\Gamma(\alpha_1-\beta_1)\Gamma(\alpha_1-\beta_2)}{\Gamma(\alpha_1-\beta_1-n)\Gamma(\alpha_1-\beta_2-n)} \tag{10}$$

$$\times {}_3F_2(-n,\ -x-\alpha_1,\ 1+\beta_1+\beta_2-\alpha_1-\alpha_2+n;\ \beta_1-\alpha_1+1,\ \beta_2-\alpha_1+1;\ 1).$$

In this form it is easily seen that the Bateman [2] polynomials are a particular case of (9): put

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = -1.$$

A recurrence relation can be given for these polynomials if we use the known contiguous function relations for ${}_pF_q$ (cf. Rainville [6]). The derivation is elementary, but messy, so we only give the result:

$$a_n p_{n+1} + b_n p_{n-1} + (c_n x + d_n) p_n = 0 \tag{11}$$

where

$$a_n = -(s+n+1)(s+2n)$$
$$b_n = n(s+2n+2)(\beta_1+n-\alpha_1)(\beta_1+n-\alpha_2)(\beta_2+n-\alpha_1)(\beta_2+n-\alpha_2)$$
$$c_n = (s+2n)(s+2n+1)(s+2n+2)$$
$$d_n = n(s+2n+2)[(\alpha_1+\alpha_2)(s+2n)+\alpha_1\alpha_2-(\beta_1+n)(\beta_2+n)]$$
$$-(s+2n)(s+n+1)[\alpha_1\alpha_2-(\beta_1+n+1)(\beta_2+n+1)]$$

and $s=\beta_1+\beta_2-\alpha_1-\alpha_2$.

A recurrence formula given for the Tchebichef polynomials by Jordan [4] is a particular case of our formula (11).

Assuming that $\alpha_2-\alpha_1$ is a positive integer, we obtain orthogonality relations for our polynomials with a weight-function which has jumps

$$\frac{\Gamma(x+\beta_1+1)\Gamma(x+\beta_2+1)}{\Gamma(x+\alpha_1+1)\Gamma(x+\alpha_2+1)}$$

at the points $x=-\alpha_1-n$, $n=0, 1 \cdots$.

Case 2: $k=1$

From equation (4) we obtain two subcases

(a) $$\frac{\rho(x)}{\rho(x-1)} = c\frac{x+\beta}{x+\alpha}$$

(b) $$\frac{\rho(x)}{(x-1)} = c(x+\gamma).$$

(a) gives

$$\rho(x) = c^x \frac{\Gamma(x+\beta+1)}{\Gamma(x+\alpha+1)} \tag{12}$$

and

$$p_n(x) = \frac{\Gamma(x+\beta+n+1)}{\Gamma(x+\beta+1)} {}_2F_1\left(-n, -x-\alpha; -x-\beta-n; \frac{1}{c}\right) \tag{13}$$

which can also be written as

$$p_n(x) = (-1)^n \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta-n)} {}_2F_1\left(-n, -x-\alpha; \beta-\alpha+1; 1-\frac{1}{c}\right). \tag{14}$$

These are polynomials derived from the Jacobi polynomials and were studied by M. Krawtchouk (cf. Szegö [7]) and J. Meixner [4].

(b) gives

$$\rho(x) = c^x\Gamma(x+\gamma+1)$$

and

$$p_n(x) = \frac{\Gamma(x+\gamma+n+1)}{\Gamma(x+\gamma+1)}\,{}_1F_1\left(-n; -x-n-\gamma; -\frac{1}{c}\right). \tag{15}$$

These polynomials are derived from the Laguerre polynomials and were first studied by Charlier (cf. Szegö [7]). The case $k=0$ does not yield anything new. We obtain

$$\rho(x) = \frac{c^x}{\Gamma(x+1)}$$

and

$$p_n(x) = c_n L_n^{x-n}(c)$$

which are again the Charlier polynomials.

We collect in a table all the polynomials represented by (2), where k is the degree of $X(x)$.

k	Polynomials
$k>2$	none
2	${}_3F_2$; special cases: Tchebichef, Bateman
1	${}_3F_1$; Krawtchouk, Meixner
	${}_1F_1$; Charlier
0	${}_1F_1$; Charlier

3. In this section we shall derive a transformation formula for a terminating ${}_3F_2$ of unit argument. This formula is one of a large group of known transformations (cf. Bailey [1]), but as we have used the result in deriving (10), we shall give the proof here.

We write

$${}_3F_2(-n, \alpha, \beta; \gamma, \delta; 1) = \frac{\Gamma(\delta)}{\Gamma(\beta)\Gamma(\delta-\beta)}\int_0^1 {}_2F_1(-n, \alpha; \gamma; t)t^{\beta-1}(1-t)^{\delta-\beta-1}dt \tag{16}$$

which can be checked easily by expanding both sides in power-series; but we have the known identity

$${}_2F_1(-n, \alpha; \gamma; t) = \frac{\Gamma(\gamma)\Gamma(\gamma+n-\alpha)}{\Gamma(\gamma+n)\Gamma(\gamma-\alpha)}\,{}_2F_1(-n, \alpha; \alpha-n-\gamma+1; 1-t).$$

We substitute this in (16), replace the variable t by $1-t$, expand and integrate term by term. We thus obtain the identity

$$
\begin{aligned}
&{}_3F_2(-n, \alpha, \beta; \gamma, \delta; 1) \\
&\quad = \frac{\Gamma(\gamma)\Gamma(\gamma + n - \alpha)}{\Gamma(\gamma + n)\Gamma(\gamma - \alpha)}\, {}_3F_2(-n, \alpha, \delta - \beta; 1 + \alpha - \gamma - n, \delta; 1).
\end{aligned} \tag{17}
$$

Using (17) over again, with the roles of γ and δ interchanged, to transform the right side of (17), we finally obtain

$$
\begin{aligned}
{}_3F_2(-n, \alpha, \beta; \gamma, \delta; 1) = {} & \frac{\Gamma(\delta)\Gamma(\gamma)\Gamma(\delta + n - \alpha)\Gamma(\gamma + n - \alpha)}{\Gamma(\delta + n)\Gamma(\gamma + n)\Gamma(\delta - \alpha)\Gamma(\gamma - \alpha)} \\
& \times {}_3F_2(-n, \alpha, 1 + \alpha + \beta - \gamma - \delta - n; \\
& \qquad 1 + \alpha - \delta - n, 1 + \alpha - \gamma - n; 1).
\end{aligned} \tag{18}
$$

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MATHEMATICAL NOTES

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ON THE APPLICABILITY OF A CERTAIN FORMULA IN THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS

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Consider the second order linear differential equation for $y(x)$,

$$Ly \equiv (p\dot{y})^{\cdot} + qy = 0 \tag{1}$$

where p is continuously differentiable and positive, and q is continuous in the open interval Δ: $a<x<b$. (A dot denotes differentiation with respect to the argument.) A well known theorem asserts the existence of a unique solution to the initial value problem $Ly=0$, $y(c)=y_0$, $\dot{y}(c)=\dot{y}_0$, $(a<c<b)$, which is continuously twice differentiable and bounded in every closed subinterval of Δ. It follows then that two linearly independent solutions of $Ly=0$ having the same properties exist.

Let $y_1(x)=u(x)$ be a nontrivial solution of equation (1) in the interval Δ. It is often taught, without further qualifications, that the expression

$$y_2(x) = u(x)\int_{\alpha_1}^{x} \frac{d\xi}{p(\xi)u^2(\xi)} \qquad (a < \alpha_1 < b), \tag{2}$$

yields the second solution linearly independent of y_1. If, however, x_0 and x_1 are two zeros of u such that $x_0<\alpha_1<x_1$ and $u(x)\neq 0$ for $x_0<x<x_1$, it may easily be verified that the formula (2) does not represent a function outside the interval $x_0<x<x_1$. Indeed,

$$\lim_{x\to x_1}\int_{\alpha_1}^{x}\frac{d\xi}{pu^2}=\infty$$

(and likewise for $x\to x_0$), since otherwise we would have $y_2(x_1)=0$ contradicting the independence of y_2 and y_1. The integrand in (2) being positive, the integral clearly diverges for $x>x_1$ which proves our assertion. A similar conclusion follows for $x<x_0$.

As an example, consider the simplest case when $p=1$ and $q=0$: $y_1=x$ is a solution and equation (2) yields the second solution $y_2=-1+x/\alpha_1$ whenever $\operatorname{sg}(x)=\operatorname{sg}(\alpha_1)$. When $\operatorname{sg}(x)=-\operatorname{sg}(\alpha_1)$, the expression (2) does not represent a function.

The object of the present note is to extend the definition of y_2 as given by (2) outside the interval $x_0<x<x_1$. Denote by x_i the zeros of $u(x)$ arranged in increasing sequence, and let $x_{i+}(x_{i-})$ be the closest zero of $\dot{u}(x)$ to the right (left) of x_i (u and $\dot{u}$ cannot vanish simultaneously). Let α_i and β_i be numbers such that

$$x_{i-} < \alpha_i < x_i < \beta_i < x_{i+}.$$

Equation (2) represents y_2 in the interval $x_0<x<x_1$. Now, integrating by parts the integral in (2), one gets

$$y_2(x) = -\frac{1}{p(x)\dot{u}(x)}+\frac{u(x)}{p(\alpha_1)u(\alpha_1)\dot{u}(\alpha_1)}+u(x)\int_{\alpha_1}^{x}\frac{qd\xi}{(p\dot{u})^2}. \tag{3}$$

Clearly, (3) is identical with (2) in the interval $x_{1-}<x<x_1$. Yet, it represents a twice differentiable solution of (1) in the larger interval $x_{1-}<x<x_{1+}$, hence an extension of (2) to the right of x_1.

To extend the validity of (3) we split the above integral in two parts, first over the interval (α_1, β_1), second, over the interval (β_1, x) and integrate the latter integral by parts, obtaining

$$y_2 = u(x)\left[\frac{1}{p(\alpha_1)u(\alpha_1)\dot{u}(\alpha_1)} - \frac{1}{p(\beta_1)u(\beta_1)\dot{u}(\beta_1)} + \int_{\alpha_1}^{\beta_1} \frac{q\,d\xi}{(p\dot{u})^2} + \int_{\beta_1}^{\alpha_2} \frac{d\xi}{pu^2}\right] + u(x)\int_{\alpha_2}^{x} \frac{d\xi}{pu^2}. \tag{4}$$

The expression (4) matches with (3) for $x_1 < x < x_{1+}$, yet it represents the solution of (1) for $x_1 < x < x_2$. Following Hadamard [1], the expression in the brackets above may be conveniently termed the finite part (FP) of

$$\int_{\alpha_1}^{\alpha_2} \frac{d\xi}{pu^2}.$$

Without further difficulty one obtains now

$$y_2 = u(x)\left[\mathrm{FP}\int_{\alpha_1}^{\alpha_k} \frac{d\xi}{pu^2} + \int_{\alpha_k}^{x} \frac{d\xi}{pu^2}\right], \qquad x_{k-1} < x < x_k, \tag{5}$$

or, simply,

$$y_2 = u(x)\left(FP\int_{\alpha_1}^{x} \frac{d\xi}{pu^2}\right), \qquad a < x < b. \tag{6}$$

Coming back to the simple example considered above, with $\alpha_1 < 0$,

$$\mathrm{FP}\int_{\alpha_1}^{\alpha_2} \frac{d\xi}{\xi^2} = \frac{1}{\alpha_1} - \frac{1}{\alpha_2}, \qquad \alpha_2 > 0,$$

so that (5) yields

$$y_2 = -1 + \frac{x}{\alpha_1} \qquad \text{for } x > 0,$$

as expected.

Our result could have been anticipated on the basis of work of Hadamard [1]; yet, to prove it, one would have to supply the derivation just carried out. There exists a more elegant method of proving our result which uses, however, the recent and still little known notion of distributions. For those familiar with the work of L. Schwartz [2] the matter may be settled in a few words. Indeed, if one considers the equation (1) as an equation for an unknown distribution y, one finds two linearly independent distributions y_1 and y_2, the latter given by (2). As a distribution, the expression (2) has a meaning in the full interval Δ, and furthermore, there corresponds to it a real function, namely the function (6).

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A THEOREM OF NESBITT

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Muir in his *Contributions to the History of Determinants*, 1900–1920, page 316, mentions a result announced by Nesbitt, but unproved, that the n-rowed determinant

$$\left| {}^{n+1}C_{2s-r} \right| = 2^{\frac{1}{2}n(n+1)},$$

where s, r denote column and row respectively, and ${}^nC_r$ has its usual meaning for $0 \leqq r \leqq n$ and is 0 in all other cases.

The form of this result suggests that it is related to the binomial expansions of $(x+y)^n \pm (x-y)^n$, and this is the basis of the two proofs given here.

Proof I.

In this proof the result is obtained entirely by "elementary" manipulations of the binomial expansion.

Let $A = P+y$ and $B = P-y$ where P, $y \neq 0$. Then

$$\frac{1}{2}(A^{2n+1} - B^{2n+1}) = R = \sum_{r=0}^{n} {}^{2n+1}C_{2r+1} P^{2n-2r} y^{2r+1} \tag{1}$$

and

$$\frac{1}{2}(A^{2n+1} + B^{2n+1}) = Q = \sum_{r=0}^{n} {}^{2n+1}C_{2r} P^{2n+1-2r} y^{2r}. \tag{2}$$

These equations can be written in the form

$$\left({}^{2n+1}C_1 - \frac{R}{yP^{2n}} \right) P^{2n+1} + \sum_{r=1}^{n} {}^{2n+1}C_{2r+1} P^{2n-2r+1} y^{2r} = 0$$

and

$$\left({}^{2n+1}C_0 - \frac{Q}{P^{2n+1}} \right) P^{2n+1} + \sum_{r=1}^{n} {}^{2n+1}C_{2r} P^{2n-2r+1} y^{2r} = 0.$$

Multiplying these by $(y^2P^{-2})^s$ for $s = 0, 1, 2, \cdots, n-1$ and re-arranging we obtain the $2n$ equations

$$\begin{aligned}
({}^{2n+1}C_1 - Ry^{-1}P^{-2n})P^{2n+1} + {}^{2n+1}C_3 P^{2n-1}y^2 + {}^{2n+1}C_5 P^{2n-3}y^4 + \cdots + {}^{2n+1}C_{2n+1} P y^{2n} &= 0 \\
({}^{2n+1}C_0 - QP^{-2n-1})P^{2n+1} + {}^{2n+1}C_2 P^{2n-1}y^2 + {}^{2n+1}C_4 P^{2n-3}y^4 + \cdots + {}^{2n+1}C_{2n} P y^{2n} &= 0 \\
(-RyP^{-2n-2})P^{2n+1} + {}^{2n+1}C_1 P^{2n-1}y^2 + {}^{2n+1}C_3 P^{2n-3}y^4 + \cdots + {}^{2n+1}C_{2n+1} P^{-1}y^{2n+2} &= 0 \\
(-Qy^2P^{-2n-3})P^{2n+1} + {}^{2n+1}C_0 P^{2n-1}y^2 + {}^{2n+1}C_2 P^{2n-3}y^4 + \cdots + {}^{2n+1}C_{2n} P^{-1}y^{2n+2} &= 0 \\
\cdots\cdots\cdots

$$\Delta_1 = \begin{vmatrix} {}^{2n+1}C_1 - Ry^{-1}P^{-2n} & {}^{2n+1}C_3 & {}^{2n+1}C_5 \cdots {}^{2n+1}C_{2n+1} \cdots & 0 \\ {}^{2n+1}C_0 - QP^{-2n-1} & {}^{2n+1}C_2 & {}^{2n+1}C_4 \cdots {}^{2n+1}C_{2n} \cdots & 0 \\ -RyP^{-2n-3} & {}^{2n+1}C_1 & {}^{2n+1}C_3 \cdots {}^{2n+1}C_{2n-1} \cdots & 0 \\ -Qy^2P^{-2n-3} & {}^{2n+1}C_0 & {}^{2n+1}C_2 \cdots {}^{2n+1}C_{2n-2} \cdots & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & \\ -Qy^{2n-2}P^{-4n+1} & 0 & 0 \cdots {}^{2n+1}C_2 \cdots {}^{2n+1}C_{2n} & \end{vmatrix} \equiv 0$$

where R, Q are defined as in (1), (2). Expanding by the first column, we obtain

$$(3) \quad N_{2n} = y^{-1}P^{1-4n} \begin{vmatrix} RP^{2n-1} & {}^{2n+1}C_3 & {}^{2n+1}C_5 \cdots {}^{2n+1}C_{2n+1} \cdots & 0 \\ QP^{2n-2}y & {}^{2n+1}C_2 & {}^{2n+1}C_4 \cdots {}^{2n+1}C_{2n} \cdots & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & \\ Qy^{2n-1} & 0 & 0 \cdots {}^{2n+1}C_2 \cdots {}^{2n+1}C_{2n} & \end{vmatrix}$$

$$= y^{-1}P^{1-4n}\Delta_2,$$

where N_{2n} is Nesbitt's determinant of $2n$ rows.

This result is true for any values of P, y not equal to 0. Write $P=y$, then

$$A^{2n+1} = 2^{2n+1}y^{2n+1}, \qquad B^{2n+1} = 0$$

and

$$R = Q = 2^{2n}y^{2n+1}.$$

With these substitutions each element of the first column of Δ_2 is $2^{2n}y^{4n}$. The other elements in Δ_2 are all ${}^{2n+1}C_{2s-r}$ and can be replaced by ${}^{2n}C_{2s-r} + {}^{2n}C_{2s-r-1}$.

If we denote rows by r and columns by c, and perform the successive operations on Δ_2 (as now re-arranged),

(1) remove the factor $2^{2n}y^{4n}$ from the first column

(2) $r_s - r_{s-1}$ for $s = 2n, \cdots 2$

(3) $c_s + c_{s+1}$ for $s = 2n-1, 2n-2, \cdots 2$

(4) expand by the first column,

we obtain

$$N_{2n} = 2^{2n}N_{2n-1}.$$

Similarly it can be shown that

$$N_{2n-1} = 2^{2n-1}N_{2n-2}.$$

Hence

$$N_n = 2^nN_{n-1} = 2^pN_2 \qquad \text{where} \qquad p = \sum_3^n r$$

But

$$N_2 = 2^3 = 2^{1+2}.$$

Hence

$$N_n = 2^q \quad \text{where} \quad q = \sum_1^n r = \tfrac{1}{2}\, n(n+1).$$

Proof II.

I am indebted to Dr. T. W. Chaundy for suggesting that this result can also be deduced from Sylvester's eliminant E, as follows:—We can show* that if

$$A(x) \equiv a_0 x^m + a_1 x^{m-1} + \cdots + a_m$$
$$B(x) \equiv b_0 x^n + b_1 x^{n-1} + \cdots + b_n,$$

then

$$E = \begin{vmatrix} a_0 & a_1 \cdots a_m & 0 & \cdots 0 \\ 0 & a_0 \cdots a_{m-1} & a_m & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_m \\ b_0 & b_1 \cdots \cdots & b_n & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_n \end{vmatrix} = a_0^n \prod_{r=1}^{m} B(\alpha_r)$$

where $\alpha_1, \alpha_2, \cdots \alpha_m$ are the zeros of $A(x)$.

To prove the required result we evaluate E when $A(x)$, $B(x)$ are particular polynomials of the form $(y+1)^n \pm (y-1)^n$, and show that E is numerically equal to N. We have to consider the odd and even cases separately.

Let

$$A(x) = \frac{(\sqrt{x}+1)^{2n+1} + (\sqrt{x}-1)^{2n+1}}{2\sqrt{x}},$$

$$B(x) = \frac{(\sqrt{x}+1)^{2n+1} - (\sqrt{x}-1)^{2n+1}}{2}.$$

Then if

$$\theta^{2n+1} = 1 \quad (\theta \neq 1), \qquad \frac{\sqrt{\alpha}+1}{\sqrt{\alpha}-1} = -\theta$$

gives the zeros of $A(x)$. Then

$$B(\alpha) = \frac{2^{2n+1}}{(\theta+1)^{2n+1}}.$$

But θ, θ^{-1} give the same α. Thus taking all the $2n$ complex θ, we obtain E^2 and not E.

* This is a simple extension of the proof given for the case $m=3$, $n=2$, by Professor E. T. Whittaker, Proc. Edinburgh Math. Soc. Series 1, Vol. 40 (1922), pp. 62–63, and quoted by Professor H. W. Turnbull, Theory of Equations, Oliver and Boyd, 1939, pp. 141–142.

i.e. $$E^2 = \prod_\theta \frac{2^{2n+1}}{(\theta+1)^{2n+1}} = 2^{2n(2n+1)}, \quad \text{since} \quad \prod_\theta (\theta+1) = 1.$$

With these values of $A(x)$, $B(x)$, $E=(-1)^{\frac{1}{2}n(n+1)}N_{2n}$ and hence $N_n = \pm 2^{\frac{1}{2}n(n+1)}$ when n is even.

To determine the sign, since the values of θ which give just n roots α are

$$\theta_r = \exp\left(\frac{2ri\pi}{2n+1}\right), \qquad r = 1, 2, \cdots, n,$$

$$B(\alpha_r) = \frac{2^{2n+1}}{(e^{2i\phi}+1)^{2n+1}} \qquad \text{where} \quad \phi = \frac{r\pi}{2n+1}$$

$$= \frac{1}{(e^{i\phi}\cos\phi)^{2n+1}}.$$

But $0<\phi<\frac{1}{2}\pi$ so $\cos\phi>0$, and the sign of $B(\alpha_r)$ is the sign of $\exp(-ir\pi)$, *i.e.* $(-1)^r$. Hence the sign of $\pi B(\alpha_r)$ is $(-1)^{\frac{1}{2}n(n+1)}$ and $N_n=2^{\frac{1}{2}n(n+1)}$, when n is even. The case when n is odd is similar. Let

$$A(x) = \frac{(\sqrt{x}+1)^{2n}+(\sqrt{x}-1)^{2n}}{2}, \qquad B(x) = \frac{(\sqrt{x}+1)^{2n}-(\sqrt{x}-1)^{2n}}{2\sqrt{x}},$$

then

$$\frac{\sqrt{\alpha}+1}{\sqrt{\alpha}-1} = \theta, \qquad \text{where} \quad \theta^{2n} = -1$$

and

$$B(\alpha) = \frac{-2^{2n}}{(\theta+1)(\theta-1)^{2n-1}}.$$

This gives $E^2=2^{2n(2n-1)}$, since $\Pi_\theta(\theta-1)=\Pi_\theta(\theta+1)=2$. Also $E=(-1)^{\frac{1}{2}n(n-1)}N_{2n-1}$, as $B(x)$ is only of degree $n-1$ and

$$B(\alpha_r) = \frac{-2^{2n}}{(e^{2i\phi}+1)(e^{2i\phi}-1)^{2n-1}}, \quad 2\phi = \frac{(2r+1)\pi}{2n}, \quad r = 0, 1, \cdots, n-1$$

$$= \frac{-2^{2n}}{(e^{i\phi})^{2n}(2i\sin\phi)^{2n-1}2\cos\phi}.$$

Hence the sign of $\Pi B(\alpha_r)$ is the same as that of $\Pi\{-e^{-(r+n)i\pi}\}$ and

$$E = (-1)^{\frac{1}{2}n(n-1)}N_{2n-1} = (-1)^{n^2+n+\frac{1}{2}n(n-1)}2^{n(2n-1)}$$

i.e. $N_n=2^{\frac{1}{2}n(n+1)}$, when n is odd, since $n(n+1)$ is even.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

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RECIPROCAL QUADRATIC EQUATIONS

M. R. SPIEGEL, Rensselaer Polytechnic Institute

A reciprocal polynomial equation of order n given by $f(x)=0$ is one for which $f(x)=\pm x^n f(1/x)$. Thus if the polynomial equation is

$$f(x)=\sum_{k=0}^{n} a_k x^k \qquad a_n \neq 0 \tag{1}$$

it follows that $f(x)=0$ is a reciprocal equation if and only if

$$a_r = a_{n-r} \quad \text{or} \quad a_r = -a_{n-r}. \tag{2}$$

It is well-known that such equations may be reduced to equations of lower degree by the substitution

$$u = x + \frac{1}{x}. \tag{3}$$

Thus for example, the reciprocal equation

$$2x^4 - 5x^3 + 6x^2 - 5x + 2 = 0 \tag{4}$$

may be written as

$$2\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) + 6 = 0, \tag{5}$$

which in turn may be reduced by the substitution (3) to the quadratic

$$2u^2 - 5u + 2 = 0. \tag{6}$$

Upon solving the quadratic (6) and using (3) the equation (4) may be solved completely.

The reciprocal equation of least order (other than the monomial equation $x \pm 1 = 0$) is a quadratic of the form

$$x^2 - rx + 1 = 0. \tag{7}$$

It is not surprising that no one (at least so far as the author knows) has applied the methods of reciprocal equations to the equation (7) since this equation is solved so simply. Nevertheless it is of interest that the method with slight variation applies.

We write (7) in the form

$$x + \frac{1}{x} = r. \tag{8}$$

Adding 2 and subtracting 2 from both sides of the equation (8) we have

$$\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 = r + 2 \tag{9}$$

and

$$\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 = r - 2. \tag{10}$$

Hence

$$\sqrt{x} + \frac{1}{\sqrt{x}} = \pm\sqrt{r + 2} \tag{11}$$

and

$$\sqrt{x} - \frac{1}{\sqrt{x}} = \pm\sqrt{r - 2}. \tag{12}$$

Multiplication of (11) and (12) yields

$$x - \frac{1}{x} = \pm\sqrt{r^2 - 4}. \tag{13}$$

Adding (8) and (13) we find

$$x = \frac{r \pm \sqrt{r^2 - 4}}{2} \tag{14}$$

as we should.

It is interesting to note that any quadratic equation can be transformed into a reciprocal quadratic equation. The general quadratic equation is given by

$$ax^2 + bx + c = 0. \tag{15}$$

Letting $x = \sqrt{(c/a)}u$ this equation becomes

$$cu^2 + b\sqrt{\frac{c}{a}}\,u + c = 0$$

or

$$u^2 + \frac{b}{\sqrt{ac}}u + 1 = 0,$$

a reciprocal quadratic equation of the form (7) with $r = -(b/\sqrt{ac})$. Hence by (14),

$$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2\sqrt{ac}}, \tag{16}$$

and since $x = \sqrt{(c/a)}u$ we have finally for the solutions of (15),

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{17}$$

as we should.

A THEOREM ON EVOLUTES

FURIO ALBERTI, University of Illinois

In this note is given what seems to be a new theorem on the notion of generating evolutes of a given twisted curve. We shall prove the following:*

THEOREM. *If a circle of second order contact at P with a twisted curve C makes a variable angle $\omega = \int_{s_0}^{s}(ds/\tau)$ with the osculating plane of C at P, as P moves along C, then the locus of the centers of these circles is an evolute.*

Proof. Consider the circle represented by $\xi^2 + (\eta - r)^2 = r^2$ and $\zeta = 0$. Rotating the axes about the ξ-axis through an angle ω, we obtain $\xi^2 + (\eta \cos \omega + \zeta \sin \omega)^2 - 2r(\eta \cos \omega + \zeta \sin \omega) = 0$. Substituting $\xi = s - \cdots$, $\eta = (s^2/2\rho) + \cdots$, $\zeta = -(s^3/6\rho\tau) - \cdots$ and demanding second order contact, we find that $r = \rho/\cos \omega$. Hence we find that the coordinates of the center of the circle are $(0, \rho, -\rho \tan \omega)$. But these are the coordinates of the points on the evolute† of the curve C, and the theorem is proved.

This theorem is, in a way, an extension of the theorem in plane geometry concerning the locus of the centers of the osculating circles to a given curve. It is also of interest to note that the theorem of this note implies that the tangents of any one evolute and the principal normal of the curve C form the angle ω, since the radius of the circle of second order contact must also be perpendicular to C and pass through P. Also note, as a corollary to the theorem, that if C is a plane curve, $1/\tau = 0$, then $\omega =$ constant; hence we obtain the twisted evolutes of the plane curve C by means of the circle of second order contact moving at a constant angle to the plane of C.

* The notation and methods follow those given in E. P. Lane's Metric Differential Geometry of Curves and Surfaces, University of Chicago Press, 1940, Chapter II.

† *Ibid.*, p. 52.

INVERSE MATRICES IN INTERPOLATION

J. P. BALLANTINE, University of Washington

1. Introduction. The standard problem of polynomial interpolation is to pass a curve, $y=P_n(x)$ through $n+1$ points. We shall take the case when $n=3$, but the reader will soon see that the method holds for any value of n. To pass a curve

$$y = A_0 + A_1x + A_2x^2 + A_3x^3 \tag{1}$$

through the four points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), it becomes necessary to solve the system of equations:

$$\begin{aligned} A_0 + A_1x_1 + A_2x_1^2 + A_3x_1^3 &= y_1, \\ A_0 + A_1x_2 + A_2x_2^2 + A_3x_2^3 &= y_2, \\ A_0 + A_1x_3 + A_2x_3^2 + A_3x_3^3 &= y_3, \\ A_0 + A_1x_4 + A_2x_4^2 + A_3x_4^3 &= y_4. \end{aligned} \tag{2}$$

The usual procedure is to establish the non-vanishing of the determinant of the matrix of coefficients:

$$X = \begin{Vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{Vmatrix}. \tag{3}$$

Then some method is given for determining the parameters A_0, A_1, A_2, A_3. Both steps can be accomplished at once by finding the inverse of the matrix X.

It is the purpose of this note to show that the desired inverse can be written by inspection.

2. The inverse. First, one new notation will be introduced.

$$(a; b, c, d, e) \quad \text{means} \quad (a-b)(a-c)(a-d)(a-e).$$

Use will be made of the well known identity:

$$x^3 - (r_1+r_2+r_3)x^2 + (r_2r_3 + r_3r_1 + r_1r_2)x - r_1r_2r_3 = (x; r_1, r_2, r_3). \tag{4}$$

Let any row of the matrix X, namely $(1, x_i, x_i^2, x_i^3)$, be multiplied (*i.e.* by the vector dot product) by the vector

$$-x_2x_3x_4, \qquad x_3x_4 + x_4x_2 + x_2x_3, \qquad -(x_2+x_3+x_4), \qquad 1.$$

The result, in view of equation (4), is simply $(x_i; x_2, x_3, x_4)$. This will vanish for $i=2$, 3, and 4, but not for $i=1$ unless one of the numbers x_2, x_3, or x_4 equals x_1.

In this manner, it is apparent that $XY=Z$, where

$$Y=\begin{Vmatrix} -(x_2x_3x_4) & -(x_3x_4x_1) & -(x_4x_1x_2) & -(x_1x_2x_3) \\ (x_3x_4+x_4x_2+x_2x_3) & (x_4x_1+x_1x_3+x_3x_4) & (x_1x_2+x_2x_4+x_4x_1) & (x_2x_3+x_3x_1+x_1x_2) \\ -(x_2+x_3+x_4) & -(x_3+x_4+x_1) & -(x_4+x_1+x_2) & -(x_1+x_2+x_3) \\ 1 & 1 & 1 & 1 \end{Vmatrix}$$

and

$$Z=\begin{Vmatrix} (x_1;\, x_2,\, x_3,\, x_4) & 0 & 0 & 0 \\ 0 & (x_2;\, x_3,\, x_4,\, x_1) & 0 & 0 \\ 0 & 0 & (x_3;\, x_4,\, x_1,\, x_2) & 0 \\ 0 & 0 & 0 & (x_4;\, x_1,\, x_2,\, x_3) \end{Vmatrix}.$$

Y requires only a little modification to be converted into the inverse of X. Its first column must be divided by the first element of Z, namely $(x_1; x_2, x_3, x_4)$, which is short for $(x_1-x_2)(x_1-x_3)(x_1-x_4)$, its second column is divided by $(x_2; x_3, x_4, x_1)$, and so forth.

The inverse of X is now in form to apply to the solution of the system of equations (2), and immediately gives the required formula for $P_n(x)$. It is interesting to note that when the inverse was obtained, it was considered as a right inverse, but when it is used, it is a left inverse. The algebra to show that it is a left inverse is not so simple, so it is fortunate that we have the theorem that a right inverse is also a left inverse.

3. Applications. When $P_n(x)$ is written as a linear combination of y_1, y_2, y_3, and y_4, the coefficients of the latter are called Lagrange coefficients. It is easily seen that the Lagrange coefficients are precisely the columns of the inverse of X.

Also with the inverse of X in mind, the coefficients of any desired integration formula can be readily obtained. Suppose the desired formula is:

$$\int_a^b f(x)dx = A_1f(x_1) + A_2f(x_2) + A_3f(x_3) + A_4f(x_4).$$

The values of a, b, x_1, x_2, x_3, and x_4 are assigned. The formula is useless until the coefficients, A_1, A_2, A_3, and A_4 are known. To find the coefficients, note that the formula must integrate correctly the functions $f(x)=1$, x, x^2, and x^3. Then

$$\begin{aligned} A_1 + A_2 + A_3 + A_4 &= (b-a), \\ A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 &= \tfrac{1}{2}(b^2-a^2), \\ A_1x_1^2 + A_2x_2^2 + A_3x_3^2 + A_4x_4^2 &= \tfrac{1}{3}(b^3-a^3), \\ A_1x_1^3 + A_2x_2^3 + A_3x_3^3 + A_4x_4^3 &= \tfrac{1}{4}(b^4-a^4). \end{aligned}$$

Here the matrix of coefficients of the unknowns is the transpose of X, so its inverse is the transpose of the inverse of X. Hence A_1 is the product of the first column of the inverse of X by the right members of the above system, namely

$$A_1 = [-(x_2x_3x_4)(b-a) + (x_3x_4 + x_4x_2 + x_2x_3)\tfrac{1}{2}(b^2 - a^2) - (x_2 + x_3 + x_4)\tfrac{1}{3}(b^3 - a^3) + \tfrac{1}{4}(b^4 - a^4)] \div (x_1; x_2, x_3, x_4).$$

As x_1 is any of the assigned values of x, the same formula gives all the coefficients.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1006. *Proposed by Leo Moser, Texas Technological College*

A wholesale jobber received radios from the manufacturer packed in two types of cartons. Each type held a different number of radios. The shipping clerk found that by judiciously selecting the right number of one or both kinds, he could fill almost any size order without opening a carton. Indeed, there were exactly six orders possible which would require a carton to be opened and repackaged.

One day the manufacturer discontinued the smaller carton and sent a new size, containing a different number of radios. The shipping clerk calculated that with the new set up there were ten different orders which would be impossible to fill without repacking. How many radios were contained in this new carton?

E 1007. *Proposed by Harvey Berry, University of Kentucky*

Prove that the G.C.D. of two numbers is equal to the G.C.D. of their sum and their L.C.M.

E 1008. *Proposed by R. H. Moorman, Tennessee Polytechnic Institute*

Through a given point draw a line cutting the sides of a given angle so as to form a triangle of given perimeter.

E 1009. *Proposed by Arthur Danzl, St. John's University*

Rectify the loop of the folium of Descartes by means of elementary functions either finitely, if possible, or approximately.

E 1010. *Proposed by Michael Goldberg, Washington, D. C.*

Given n equal unit squares, dissect each of these in exactly the same way with straight cuts into $p(n)$ parts such that the n $p(n)$ pieces may be assembled to form a square of edge $\sqrt{n}$. Investigate $k(n)$, the minimal values of $p(n)$, and show that $k(n) \leqq 5$ for all n.

SOLUTIONS

The Wine Cask Problem

E 962 [1951, 260]. *Proposed by B. H. Brown, Dartmouth College*

A cask of unit volume is full of wine. A man withdraws from the cask an amount $a (0 < a < 1)$, and then adds amount a of *water*, which is assumed to mix perfectly with the wine. He continues this process b times. He then withdraws amount a and adds amount a of *wine*, continuing this process b times.

(1) Derive a general formula for the amount of wine in the cask after the $2b$ operations.

(2) If $1/a = A$ is an integer, and if B is that corresponding (integral) value of b for which the eventual wine-content is a minimum, so determine A and B that the wine-content is (a) the minimum minimorum (b) the maximum minimorum.

(3) If $A = 51$, find B and the wine-content.

(4) Find $\lim_{A \to \infty} (B/A)$.

Solution by A. H. Payne, University of North Carolina. It follows from an induction on b that

(1) *The wine-content, W, after $2b$ operations, is given by*

$$W = 1 - (1 - a)^b + (1 - a)^{2b},$$

where use is made of the fact that after b operations (of removing wine) the wine-content is equal to $(1-a)^b$. This last relation itself follows from an easy induction.

Now

$$\partial W/\partial b = (1 - a)[2(1 - a)^b - 1] \log (1 - a).$$

Since $0 < a < 1$, a zero of $\partial W/\partial b$ requires $(1-a)^b = 1/2$. If b is considered a real continuous variable this last equation determines, for each a (or A) the b which enables W to attain its minimum value of $3/4$. Denote this critical value of b by $b(A)$. Then

$$b(A) = -\frac{\log 2}{\log (1 - a)} = \frac{\log 2}{\log A - \log (A - 1)}.$$

But only for $A=2$ is $b(A)$ an integer. Therefore, for integral b,

(2a) *The minimum minimorum of wine-content is* 3/4 *and occurs at* $(A, B)=(2, 1)$.

Since W decreases with increasing b for $0<b<b(A)$, and increases for increasing $b>b(A)$, B must be one of the integers adjacent to $b(A)$. Set $p=b(A)-[b(A)]$. Then the integers

$$B' = b(A) - p \quad \text{and} \quad B' + 1$$

are the two possibilities for B. Now denote the wine-content at (A, b) by $W(A, b)$, and let $D(A, b)$ denote the difference between the wine-content at b, for fixed A, and the attainable (for real b) minimum of 3/4. That is, $D(A, b)=W(A, b)-3/4$. In this notation

$$W(A, B') = 1 - (1 - a)^{b(A)-p} + (1 - a)^{2b(A)-2p}, \qquad (0 < p < 1)$$

and, since $(1-a)^{b(A)}=1/2$,

$$D(A, B') = 1/4 - (1/2)(1 - a)^{-p} + (1/4)(1 - a)^{-2p}.$$

Similarly,

$$D(A, B' + 1) = 1/4 - (1/2)(1 - a)^{1-p} + (1/4)(1 - a)^{2-2p}.$$

Now $D(A)=\min\{D(A, B'), D(A, B'+1)\}$ is the difference between the minimum obtainable with integral b (*i.e.* B) and 3/4, obtainable with real b, for a given A.

(2b) *The maximum minimorum of wine-content is* 0.75390625 *and occurs at* $(A, B)=(4, 2)$.

Proof: By calculation, $D(4)=1/256$. It will be shown that for $A\neq 4$, $D(A)<1/256$. This is accomplished by showing that, for $A\neq 4$, both

$$D(A, B') - 1/256 < 0 \quad \text{and} \quad D(A, B' + 1) - 1/256 < 0.$$

In the first case we wish to show that

$$D(A, B') - 1/256 = 63/256 - (1/2)(1 - a)^{-p} + (1/4)(1 - a)^{-2p} < 0.$$

Denote by $U(p)$ the result of multiplying the left member by $256(1-a)^{2p}$. Then

$$U(p) = 63(1 - a)^{2p} - 128(1 - a)^{p} + 64,$$

and it is sufficient to show that $U(p)<0$. This follows at once from: (i) $U(0)=-1$; (ii) $U(1)<0$ if $a<1/9$; (iii) $dU/dp>0$ (hence $U(p)$ is increasing as p increases from 0 to 1). This means that

$$D(A, B') < 1/256 \quad \text{if} \quad A > 9.$$

Similarly

$$D(A, B' + 1) < 1/256 \quad \text{if} \quad A > 8.$$

In order to complete the proof of (2b) it is necessary only to test $D(A)$ by numerical calculation for $A=3, 5, 6, 7, 8, 9$.

Since $D(A, B') \leqq D(A, B'+1)$ implies

$$p = b(A) - [b(A)] \leqq \frac{\log (2-a) - \log 2}{\log (1-a)},$$

it follows that

$$[b(A)] \geqq \frac{\log (2A-1) - \log A}{\log A - \log (A-1)}.$$

Therefore B can be determined as follows:

$$B = B', \qquad B = \text{either } B' \text{ or } B'+1, \qquad B = B'+1,$$

according as

$$[b(A)] \gtreqless \frac{\log (2A-1) - \log A}{\log A - \log (A-1)}.$$

(3) *If* $A=51$, *the above criterion gives* $B=35$; *and, therefore,*

$$W(51, 35) = 1 - (50/51)^{35} + (50/51)^{70} \doteq 0.750000000762.$$

(4) *Finally,*

$$\lim_{A\to\infty} (B/A) = \lim_{A\to\infty} (b(A)/A) = \lim_{A\to\infty} \frac{(\log 2)/A}{\log A - \log (A-1)} = \log 2.$$

Also solved by A. L. Epstein, Prasert Na Nagara, W. O. Pennell, and the proposer.

A Minimum Right Triangle

E 976 [1951, 491]. *Proposed by R. D. Stalley, University of Arizona*

What is the least area of a right triangle which can be placed on a rectangular coordinate system so that no side is parallel to a coordinate axis and so that the coordinates of its vertices and the lengths of its sides are integral?

Solution by R. Z. Vause, Jr., Clemson Agricultural College. Let a and b denote the legs of the triangle and c the hypotenuse. Let a_x, a_y and b_x, b_y denote the projections of a and b on the coordinate axes. Then

$$a_y/b_x = a_x/b_y = A/B, \qquad (A, B) = 1,$$

and there exist non-zero m and n such that

$$a_y = mA, \qquad b_x = mB, \qquad a_x = nA, \qquad b_y = nB.$$

The area of the triangle is given by

$$K = AB(m^2 + n^2)/2.$$

Since

$$a^2 = A^2(m^2 + n^2), \qquad c^2 = (A^2 + B^2)(m^2 + n^2),$$

it follows that A, B, m, n must be chosen so that m^2+n^2 and A^2+B^2 are both squares. There is no solution smaller than $A=m=3$, $B=n=4$, whence $K=150$. Such a triangle is the one having its vertices at $(0, 9)$, $(12, 0)$, $(-12, -7)$.

Also solved by Leon Bankoff, J. H. Braun, F. F. Dorsey, R. T. Hood, Sidney Kravitz, Josef Langr, Prasert Na Nagara, Leola Odland, C. S. Ogilvy, Elijah Swift, G. W. Walker, and the proposer.

A Family of Ellipses

E 977 [1951, 491]. *Proposed by Jose Gallego-Diaz, Madrid, Spain*

Let OA be a radius of a circle with center O. Produce OA its own length to B. Through A draw a horizontal line and on it mark off, in each direction from A, $AM=AN=OA$. If OB and MN are taken as the equal conjugate diameters of an ellipse, find the loci of the vertices and the foci of this ellipse as the radius OA varies. Also find the envelope of the major axis of the ellipse as OA varies.

Solution by Roscoe Woods, State University of Iowa. Without loss of generality, let the fixed circle be taken as the unit circle. Let the point A be given by the coordinates $(\cos 2\theta, \sin 2\theta)$. The equations of the bisectors of the angles formed by the lines OB and MN are

$$x \sin\theta - y\cos\theta + \sin\theta = 0 \tag{1}$$

and

$$x\cos\theta + y\sin\theta - \cos\theta = 0. \tag{2}$$

The major axis of the ellipse through O, B, M, N which has OB and MN as equal conjugate diameters will lie on line (1) when A is any point on the unit circle in the first and fourth quadrants, and on line (2) when A lies in the second and third quadrants. The lengths of the semimajor and semiminor axes of the ellipse are readily found to be $\sqrt{2}\cos\theta$ and $\sqrt{2}\sin\theta$ respectively. The equation of the ellipse is then readily found to be

$$\frac{(x\cos\theta + y\sin\theta - \cos\theta)^2}{2\cos^2\theta} + \frac{(x\sin\theta - y\cos\theta + \sin\theta)^2}{2\sin^2\theta} = 1. \tag{3}$$

It should be pointed out that the locus of (3) is a circle when $\theta=45°$ or $135°$, and a degenerate or flat ellipse when $\theta=0°$ or $90°$.

If the parameter θ be eliminated between (1) and (3), and between (2) and (3), the equations of the loci of the extremities of the axes are readily found to be the four circles

$$x^2 + y^2 \pm \sqrt{2}x \pm \sqrt{2} - 1 = 0.$$

The foci of the system of ellipses in (3) are the intersections of line (1) and the circle

$$(x - \cos 2\theta)^2 + (y - \sin 2\theta)^2 = 2 \cos 2\theta$$

when A lies in the first and fourth quadrants. When A lies in the second and third quadrants, the foci of the system are the intersections of line (2) and the circle

$$(x - \cos 2\theta)^2 + (y - \sin 2\theta)^2 = -2 \cos 2\theta.$$

The result of eliminating θ between the two pairs of equations gives, as the loci of the foci, the two circles

$$x^2 + y^2 \mp 2x - 1 = 0.$$

Obviously the envelopes of the lines on which the major axes of the ellipses lie are the two points $(\pm 1, 0)$.

Also solved by Pompeys Alonso.

A Lemma Used by Hansraj Gupta

E 978 [1951, 491]. *Proposed by J. M. Kingston, University of Washington*

If $[x]$ denotes the integral part of x and if j is any positive integer, show that

$$\sum_{i=0}^{[j/3]} [(j - 3i)/2] = [(j^2 + 2j + 4)/12].$$

Solution by S. T. Thompson, Plattsburg, N. Y. Set

$$F(j) = \sum_{i=0}^{[j/3]} [(j - 3i)/2] - [(j^2 + 2j + 4)/12].$$

One easily verifies that $F(j) = 0$ for $j = 0, 1, 2, 3, 4, 5$. Now suppose $F(n) = 0$. Then

$$\begin{aligned} F(n+6) &= \sum_{i=0}^{[n/3]+2} [(n + 6 - 3i)/2] - [\{(n+6)^2 + 2(n+6) + 4\}/12] \\ &= \sum_{i=-2}^{[n/3]} [(n - 3i)/2] - [(n^2 + 2n + 4)/12] - n - 4 \\ &= 4 + [n/2] + [(n+1)/2] + \sum_{i=0}^{[n/3]} [(n - 3i)/2] \\ &\quad - [(n^2 + 2n + 4)/12] - n - 4 = 0. \end{aligned}$$

This proves the theorem.

Also solved by J. H. Braun, C. V. Fronabarger, Sidney Kravitz, Prasert Na Nagara, C. F. Pinzka, R. Z. Vause, Jr., W. D. Wood, and the proposer.

Scoring a Test

E 979 [1951, 491]. *Proposed by E. S. Keeping, University of Alberta*

A student is asked on a test to match n given historical events against n different given dates, assigning a date to each event. He is sure of k dates, but is completely ignorant of the others and assigns them at random. Show that a fair method of scoring would be to deduct 1 from the number of correct matches.

Solution by Frank Herlihy, Comstock, Michigan. Let $n-k=r$ be the number of dates to be assigned at random. Let the score of the guesses be

$$s = x_1 + x_2 + \cdots + x_r,$$

where each x_i has the value 0 if it is not a coincidence and 1 if it is. The mathematical expectation of the score will then be

$$E(s) = E(x_1) + E(x_2) + \cdots + E(x_r).$$

The probability of a coincidence in the ith place will be $1/n$, whence $E(x_i) = (1/n)1 = 1/n$. Therefore $E(s) = (1/n)n = 1$. Thus the mathematical expectation gives us an extra score of 1 point from the random guesses. If the student knew all but one coincidence he would be graded correctly, but if he knew all he would be cheated of one point.

This solution is shown on page 169 of Uspensky's *Introduction to Mathematical Probability.*

Also solved by J. H. Braun, F. F. Dorsey, T. L. Jordan, Jr., C. F. Pinzka, W. R. Van Voorhis, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey.

PROBLEMS FOR SOLUTION

4474 [1952, 109]. *Correction.*

Equation (3) should be changed to read:

$$|Z_1 Z_2 \cdots Z_k|^2 \leqq \tfrac{1}{2}(B^2 + \sqrt{B^4 - 4|a_0|^2}), \quad (1 \leqq k \leqq n). \tag{3}$$

4478. *Proposed by Albert Wilansky, Lehigh University*

A basis in a Banach space B is a set of elements $\{x_n\}$ such that every element of the space is a unique, finite or infinite, linear combination of elements of the set.

(1) Prove that the x_n are isolated.

(2) If $\sum x_n$ is an element of B, show that there is a divergent sequence $\{s_n\}$ of numbers such that $\sum s_n x_n$ is an element of B.

4479. *Proposed by D. J. Newman, Harvard University*

Suppose that all the series

$$\sum_{n=0}^{\infty} \alpha_n, \quad \sum_{n=0}^{\infty} \alpha_n^2, \quad \sum_{n=0}^{\infty} \alpha_n^3, \cdots$$

converge to zero, where the α_n are complex numbers. Does it follow that all the $\alpha_n = 0$?

4480. *Proposed by Rufus Isaacs, the Rand Corporation, Santa Monica, California*

Given a closed solid polyhedron with faces which are equilateral triangles, we can construct another by the following process. Join the mid-edge points of each face, dividing it into four triangles. On each face place externally a regular tetrahedron having the central triangle for a base.

If we start with a regular tetrahedron and repeat this process, show that the limiting figure is a cube less a certain point set which is non-enumerable, yet of measure zero.

4481. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a system of numeration with base B, there are n one-digit numbers less than B whose cubes have $B-1$ as the unit's digit. Determine the relation between B and n.

4482. *Proposed by Harold Shapiro, Massachusetts Institute of Technology*

Let $n_1 < n_2 < \cdots$ be any sequence of positive integers (finite or infinite), and let $T_{n_1 n_2 \ldots} x$ denote a transformation which alters precisely the n_1, n_2, $\cdots$ digits in x, where $x (0 < x < 1)$ is expanded in dyadic notation. Show that $T_{n_1 n_2 \ldots}$ is measure-preserving on subsets of (0, 1).

SOLUTIONS

Generalization of Euler's Differential Equation

4406 [1950, 490]. *Proposed by H. J. Zimmerberg, Rutgers University*

Show that the most general linear differential equation

$$\sum_{i=0}^{n} a_i [p(x)]^i D^i y = f(x),$$

where $p(x)$ is of class C' and the a_i are constants with $a_n \neq 0$, which is reducible to the case of constant coefficients by a transformation of the independent variable is the generalized form of Euler's (sometimes called Cauchy's) equation, wherein $p(x)$ is linear in x.

Solution by the Proposer. Consider the change of independent variable $x=\phi(u)$, where ϕ is of class $C^{(n)}$ and $\phi'(u)\neq 0$. If $D^i\equiv d^i/dx^i$ and $\mathcal{D}^i\equiv d^i/du^i$, it then follows at once by mathematical induction that

$$D^r = (1/\phi')^r(\mathcal{D} - (r-1)\phi''/\phi')(\mathcal{D} - (r-2)\phi''/\phi') \cdots (\mathcal{D} - \phi''/\phi')\mathcal{D}$$

for arbitrary positive integers r. Substituting and equating the coefficients of $\mathcal{D}^n$ and $\mathcal{D}^{n-1}$ to constants, we obtain

$$\frac{a_n p^n[\phi(u)]}{\phi'^n} \equiv c_n \neq 0, \tag{1}$$

$$\frac{a_{n-1}p^{n-1}[\phi(u)]}{\phi'^{n-1}} - \frac{n(n-1)}{2}\,\frac{a_n p^n[\phi(u)]\phi''}{\phi'^{n+1}} \equiv c_{n-1}. \tag{2}$$

As $\phi'(u)=(a_n/c_n)^{1/n}p[\phi(u)]$ from (1), relation (2) reduces to $\phi''/\phi'=$constant. However, $\phi''(u)=(a_n/c_n)^{1/n}((d/dx)p[\phi(u)])\phi'(u)$ and, therefore, $p'(x)=$constant. Hence $p(x)$ is linear in x.

Also solved by M. S. Klamkin.

Conics Circumscribed about the Faces of a Tetrahedron

4407 [1950, 490]. *Proposed by N. A. Court, University of Oklahoma*

If four conics are circumscribed about the faces of a tetrahedron (T) in such a way that at each vertex of (T) the three tangents to the three conics passing through that vertex are coplanar, then the four conics lie on a quadric surface.

Solution by the Proposer. Let (q_a), (q_b), (q_c), (q_d) be four conics circumscribed about the faces BCD, CDA, DAB, ABC of $(T)\equiv ABCD$ so that the triads of tangents to these conics at the points A, B, C, D lie in the planes α, β, γ, δ, respectively. The three tangents to (q_d) at the vertices A, B, C of triangle ABC meet the respectively opposite sides BC, CA, AB in the three points P, Q, R which are collinear. (The line PQR may be called the Lemoine axis of the triangle ABC for the conic considered.) The tangent AP lies in the plane α, whence P lies on the line of intersection (α, DBC) of the two planes α and DBC. Similarly for the points Q, R. Thus the lines of intersection of the three pairs of planes α, DBC; β, DCA; γ, DAB are met by the line PRQ which lies in the plane ABC and therefore also meets the line (δ, ABC).

Considering the other faces of (T) in the same manner we conclude that the lines of intersection of corresponding faces of the two tetrahedrons (T) and $\alpha\beta\gamma\delta$ are met by the Lemoine axes of the triangles BCD, CDA, DAB, ABC for the conics (q_a), $\cdots$. Hence the two tetrahedrons are skewly perspective and, therefore, a quadric surface (Q) may be inscribed in the tetrahedron $\alpha\beta\gamma\delta$ touching its faces at the points A, B, C, D, respectively (H. F. Baker, *Principles of Geometry*, Cambridge, 1923, vol. 3, p. 53, §14.)

The plane ABC cuts the quadric (Q) along a conic (p) passing through the points A, B, C and having for tangents at these points the traces of the planes α, β, γ in the plane ABC. (p) is therefore identical with (q_d), that is, (q_d) lies on the quadric (Q). Similarly for the other three conics. Hence the proposition.

Note. The four Lemoine axes considered form a hyperbolic group supplementary to the hyperbolic group formed by the lines of intersection of the corresponding faces of the two tetrahedrons. By reversing the steps of the above proof we obtain the theorem: *If four conics are circumscribed about the faces of a tetrahedron so that the Lemoine axes of the triangles for the respective conics form a hyperbolic group, the four conics lie on a quadric surface.*

Sign of $\Gamma^{(n)}(1)$

4408 [1950, 490]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute*

Prove that $\Gamma^{(n)}(1)$ is positive or negative according as the integer n is even or odd.

Solution by C. F. Pinzka, Xavier University, Cincinnati, Ohio. Defining

$$\Gamma(x) = \int_0^\infty e^{-y} y^{x-1} dy, \qquad x > 0,$$

the conditions for differentiating under the integral sign are satisfied and

$$\begin{aligned}\Gamma^{(n)}(1) &= \int_0^\infty e^{-y} (\log y)^n dy \\ &= \int_0^1 e^{-y} (\log y)^n dy + \int_0^1 (-1)^n e^{-1/y} y^{-2} (\log y)^n dy \\ &= \int_0^1 [y^2 + (-1)^n e^{y-1/y}] e^{-y} y^{-2} (\log y)^n dy.\end{aligned}$$

When n is even, the integrand is always positive, whence the integral is positive. When n is odd, the integrand has a sign opposite to that of the expression in brackets, which becomes $y^2 - e^{y-1/y}$, and which can be proved positive for $0 < y < 1$ as follows:

$$y^2 = e^{2 \log y} = e^{2[(y-1)-(y-1)^2/2+\cdots]} > e^{-y^2+4y-3} > e^{y-1/y}.$$

Also solved by M. S. Klamkin, J. G. Millar, C. S. Ogilvy, O. E. Stanaitis, R. E. Wild, A. G. Clark's Class in Probability Theory, and the proposer.

Convergent and Divergent Series

4409 [1950, 490]. *Proposed by Fritz Herzog, Michigan State College*

Given a power series $f(x) = a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ with real coefficients. It

is assumed that at least one of the coefficients a_n, $n \geqq 2$, is different from zero and that the radius of convergence R is positive or infinite. Show that there exist constants u_n, $0 < u_n < R$, such that $\sum_n (-1)^n u_n$ converges but $\sum_n (-1)^n f(u_n)$ diverges.

Solution by G. G. Lorentz, University of Toronto. Let a_p be the first coefficient a_n, $n \geqq 2$, with $a_n \neq 0$. Define

$$u_{4k+j} = C_j k^{-1/p}, \qquad k = K, K+1, \cdots; j = 0, 1, 2, 3;$$
$$C_0 = 1, \quad C_1 = 2, \quad C_2 = 3, \quad C_3 = 2;$$

and K is an integer such that $K > (3/R)^p$. Then

$$\sum_{j=0}^{3} (-1)^j u_{4k+j} = k^{-1/p}(C_0 - C_1 + C_2 - C_3) = 0,$$

and therefore $\sum_n (-1)^n u_n$ converges with sum 0.

On the other hand, for small x, $f(x) = a_1 x + a_p x^p + O(x^{p+1})$, and we obtain for large k,

$$\sum_{j=0}^{3} (-1)^j f(u_{4k+j}) = a_p k^{-1}(C_0^p - C_1^p + C_2^p - C_3^p) + O(k^{-(p+1)/p}).$$

Since $C_0^p - C_1^p + C_2^p - C_3^p \geqq 1$ for $p = 2, 3, \cdots$, and the series with the term $k^{-(p+1)/p}$ is convergent, we see that $\sum_n (-1)^n f(u_n)$ behaves like the series $\sum k^{-1}$, that is, it is divergent.

Also solved by the proposer, and by another contributor who did not give his name.

Tchebychef Polynomials

4410 [1950, 564]. *Proposed by C. D. Olds, San Jose State College, California*

Show that, for every positive integer n, there is a polynomial $f(x)$ of degree n and a related polynomial $\phi(x)$ which satisfy

$$[f(x)]^2 - 1 = (x^2 - 1)[\phi(x)]^2.$$

I. *Solution by W. V. Parker, Alabama Polytechnic Institute.* Write

$$(x + \sqrt{x^2 - 1})^n \equiv f(x) + \sqrt{x^2 - 1}\,\phi(x),$$

where $f(x)$ and $\phi(x)$ are uniquely defined polynomials. Then

$$(x - \sqrt{x^2 - 1})^n \equiv f(x) - \sqrt{x^2 - 1}\,\phi(x),$$

and since the product of the two left members is 1, we have

$$1 = [f(x)]^2 - (x^2 - 1)[\phi(x)]^2 \tag{1}$$

which is the required relation. Explicit expressions for $f(x)$ and $\phi(x)$ are

$$f(x) = \frac{(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n}{2},$$

$$\phi(x) = \frac{(x + \sqrt{x^2 - 1})^n - (x - \sqrt{x^2 - 1})^n}{2\sqrt{x^2 - 1}}.$$

A polynomial $f(x)$ of degree n which satisfies (1) may be readily shown to be unique except for sign. Since $f(x)$ and $\phi(x)$ are relatively prime we obtain by differentiation that $n\phi(x) = f'(x)$. The polynomial $f(x)$ satisfies the differential equation

$$(x^2 - 1)f''(x) + xf'(x) - n^2 f(x) = 0$$

for positive integral values of n.

II. *Solution by J. B. Kelly, Institute for Advanced Study.* Set $x = \cos\theta$. (It is not necessary to restrict x by limiting θ to real values.) Put $f(x) = \cos n\theta$, $\phi(x) = \sin n\theta / \sin\theta$. It is easily established by induction that $f(x)$ and $\phi(x)$ are polynomials. Further, the proposed relation is satisfied identically.

III. *Solution by A. B. Boggs, Michigan College of Mining and Technology.* In his article, *Sur les questions de minima qui se rattachement à la représentation approximative des fonctions*, Tchebychef considers the equation

$$[f(x)]^2 - L^2 = (x^2 - h^2)[\phi(x)]^2.$$

See pp. 297–300 of *Oeuvres de Tchebychef*, v. I, 1899. The solution to the present problem is obtained at once with $L = h = 1$.

Also solved by H. L. Alder and D. A. Norton, Norman Anning, A. B. Boggs (an independent solution), D. H. Browne, L. Carlitz, S. H. Eisman, J. E. Freund, P. M. Hummel, M. S. Klamkin, Roger Lessard, G. Lumer, Fred Marer, S. T. Parker, C. F. Pinzka, G. F. Rose, R. D. Schafer, O. E. Stanaitis, Hans Stetter, Morgan Ward, and the Proposer.

Editorial Note. If f_n and ϕ_n are polynomials having positive leading coefficient and satisfying the given relation, so are

$$\begin{aligned} f_{n+1} &= xf_n + (x^2 - 1)\phi_n \\ \phi_{n+1} &= f_n + x\phi_n, \end{aligned} \tag{2}$$

as is easily verified directly. Upon solving (2) we have

$$\begin{aligned} f_n &= xf_{n+1} - (x^2 - 1)\phi_{n+1} \\ \phi_n &= -f_{n+1} + x\phi_{n+1}, \end{aligned} \tag{3}$$

and also

$$f_n + f_{n+2} = 2xf_{n+1}, \qquad \phi_n + \phi_{n+2} = 2x\phi_{n+1}. \tag{4}$$

Suppose now that f_{n+1} and ϕ_{n+1} are any pair of polynomials which satisfy (1) with $f_{n+1} = ax^{n+1} + bx^n + cx^{n-1} + \cdots$, $a > 0$. Then from (1)

$$\phi_{n+1} = ax^n + bx^{n-1} + (c + \tfrac{1}{2}a)x^{n-2} + \cdots .$$

Now (3) gives $f_n = \frac{1}{2}ax^n + \cdots$, $\phi_n = \frac{1}{2}ax^{n-1} + \cdots$ which satisfy (1), and by repeated application of (3) there are obtained pairs of polynomials for each lower value of the subscript, until we reach for $n=1$, $f_1 = x$ and $\phi_1 = 1$, since these are the only positive solutions, with positive leading coefficient, with f_1 of the first degree. It follows that all polynomial solutions are obtained, except for sign, from $f_1 = x$ and $\phi_1 = 1$ by repeated application of (2).

With the substitutions suggested in the second solution above, equations (2), (3), (4) are immediately equivalent to familiar trigonometric identities, as are also the following:

$$\phi_n f_{n-1} - f_n \phi_{n-1} = 1, \qquad f_n f_{n-1} - x = (x^2 - 1)\phi_n \phi_{n-1}.$$

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

1. *Essentials of College Algebra.* By J. B. Rosenbach and E. A. Whitman. New York. Ginn and Company, 1951. 10+322+30 pages. $3.00.
2. *College Algebra.* By H. L. Rietz, A. R. Crathorne, and J. W. Peters. New York. Henry Holt and Company, 1951. 17+387 pages.
3. *College Algebra.* By H. K. Fulmer and Walter Reynolds. New York. Ginn and Company, 1951. 6+204+14 pages.
4. *Intermediate College Algebra.* By E. M. J. Pease. New York. Prentice-Hall, Inc., 1950. 7+420+36 pages. $2.85.
5. *Intermediate Algebra.* By P. K. Rees and F. W. Sparks. New York. McGraw-Hill Book Company, 1951. 8+328 pages. $3.25.
6. *Elements of Algebra.* By L. C. Peck. New York. McGraw-Hill Book Company, 1950. 13+230 pages. $2.75.
7. *Practical Mathematics, Part II, Algebra with Applications.* By C. I. Palmer and S. F. Bibb. New York. McGraw-Hill Book Company, Inc., 1950. 12+252 pages. $2.20.
8. *Algebra for Commerce and Liberal Arts.* By A. K. Bettinger and W. A. Dwyer. New York. Pitman Publishing Corporation, 1951. 11+225 pages.

The content material of these books varies from the older habit of including partial fractions, limits, and infinite series as in (2) to the modern habit of more careful explanations with sacrifice of content, as in (6). (1), (2), (3) and (4) give certain coverage to such topics as simultaneous quadratics, progressions,

logarithms, binomial theorem, theory of equations, and complex numbers, while (1), (2) and (3) also include determinants and inequalities. Mathematical induction is covered by (2) and (3) while (8) includes a brief useful chapter on the slide rule. (4) has the unique feature of certain clever tests at the end of the chapters.

An impartial reader notes that (1), (2) and (3) are strong in content material of the classical type. (4) and (5) are slanted more for the intermediate (as their titles clearly imply) algebra student, while (6) is a jewel for the belated college student. (7) and (8) fulfill different purposes (see titles).

(1) "is considerably shorter than the authors' *College Algebra*, Third Edition, the same standards in fullness of explanation have been maintained," according to the preface. The usual topics in college algebra for the past two decades are found in this text. Common student mistakes are highlighted by 'Warnings' distributed throughout the book showing both the correct and incorrect procedures, such as $f(x)$ does not mean f times x. Answers to odd problems, as well as certain appropriate tables, combine to make the book a useful educational asset.

First published in 1909, and revised every ten years except 1949, (2) is well known for the breadth of topics covered and for its careful exposition of the facts. The first chapter has been rewritten to emphasize the number system. The preface reminds the reader that "most of the exercises and problems have been changed and an effort has been made to arrange them in order of increasing difficulty." Brief tables of powers, roots, reciprocals, and logarithms are noted in the back of the book.

(3) reminds one of the scholarly books of the last two decades as far as content is concerned, with commendable effort to define for the student his vocabulary of integer, term, monomial, binomial, product, factor, coefficient, and exponent in the first chapter. Certain elementary concepts have been amplified, as "Thus, in the treatment of fractions the idea of simple indeterminate forms is explained," (preface). Even mathematical induction appears in this text, which is a small book, as far as exterior is concerned. Answers to all problems, plus the brief logarithm table combine to make this book one of interest to the reader.

The author of (4) says that "the fundamentals of arithmetic and algebra are covered to such an extent that a good student may use this text for his first course in algebra." One must admit that the tests arranged at the end of the chapters should speed the student along so that he might also cover a great many of the solid topics, such as simultaneous quadratics, theory of equations, and binomial theorem. While no tables appear, yet a good index and answers to even problems assure the student of an acceptable text for its purpose.

(5) "is designed for the student who has had only one year of high school algebra" and "the problems are arranged in groups of four on about the same level of difficulty", the authors state. Timely definitions of such terms as integer, rational number, algebraic expression, term, and the stating of axioms concern-

ing equations help to remind present day mathematics teachers that their students have limited mathematical heritage. On page 40 one notices a pleasant effort to explain that zero times a finite number is zero. Answers to all even problems and half of the odd problems appear. Topics such as systems of quadratic equations in two variables, logarithms, progressions, and binomial theorem, make this text suitable for a large group of students.

(6), "written primarily for college students who have had no algebra in high school and thus presupposes no mathematical background other than ordinary arithmetic," according to the author, accomplishes its purpose, as far as explanations are concerned. Even though the identity sign is not used, great emphasis is placed upon such operations; later the conditional equation appears on page 63. Elementary work on translations from English to algebra on page 76 of Chapter V, interesting facts concerning the presence of zero in the denominator in next chapter, and careful illustrations on page 97 concerning reducing a fraction to its lowest terms help to interest the reader. The unusual practice of omitting a numerator or a denominator in some problems reminds one of examination questions instead of text book problems. Factoring and graphs get painstaking explanations while the exhibit of the square root of 2 is a welcome addition for an introduction to quadratic equation study. Answers to odd problems, as well as additional answers in the last chapter are given. Including a chapter on complex numbers in such a book increases the stature of this publication.

Like (2), (7) is the fifth edition. It is a useful book known since its first edition in 1912. It is the second part of a series of four books in practical mathematics. While this text does include progressions (this latter a new chapter), yet its mission is emphasized in the more elementary processes of algebra. Hence, no tables are available, but answers to most all problems, as well as a brief index, appear.

The inclusion of brief chapters in (8) on curve fitting, extraction of square roots (it does have a logarithm chapter also), and slide rule, in addition to most of the usual topics found in good college algebra books is noted. Appropriate topics of percentage, markup, and markdown are sandwiched into the chapter on ratio, proportion, and variation. Answers to odd problems are given.

W. R. HUTCHERSON
University of Florida

The Teaching of Mathematics. By D. R. Davis. Cambridge, Mass., Addison-Wesley Press, Inc., 1951. xv+415 pp.

Into this text, intended for a one semester course, the author has attempted to put what he considers the essentials of effective teaching; namely, (1) teaching techniques, procedures, and trends discussed in relation to the subject matter of mathematics and with especial emphasis on means of creating interest, (2) a systematically organized knowledge of the subject matter, plus (3) practical suggestions for the supervision of mathematics instruction and for the

preparation and out-of-class duties of mathematics teachers.

The distribution of time to these topics is as follows: topic (1) is treated in chapters 1–5, (2) in chapters 7–14, (3) in chapters 15, 16. This leaves Chapter 6 "Organization and Treatment of Subject Matter" to serve in part a unifying role and in part as the place where the author re-emphasizes his own themes in and philosophy of mathematical instruction. These seem to be (a) the importance of sequence, order, and logic in the organization and teaching of mathematics, (b) the need for a teacher to understand and present in a broad and general way the postulational nature of the foundations of mathematics, but (c) to realize simultaneously that complete rigor is not suited to elementary instruction where a psychological organization must also be planned. In this connection the author stresses (d) the importance of the use of *analysis*, and the role of intuition and discovery in both mathematics and teaching, and (e) the importance of generalization, of stressing fundamental concepts rather than memorized formulas, and relationships and fusion rather than watertight compartmentalization.

All of these "themes" are excellent in the reviewer's opinion. Among the strongest points of the book are its detailed examples of the analysis of theorems and of the construction of teaching units ending with generalizations of earlier theorems. These are in line with the author's assertion that what most teachers want is "practical assistance" in making teaching theories concrete. Similarly concrete and practical are many of his suggestions for rationalizing processes and enriching teaching, such as the graphical interpretation of the solution of simultaneous linear equations by addition and the discussion of Pythagorean numbers. However, the reviewer wondered if the rather lengthy syllabi or course outlines for all grades found in chapters 7–12 were not in the first place arbitrary and unnecessarily detailed while on the other hand the discussion of such topics as the conic sections appears too sketchy for persons with inadequate background in freshman mathematics and too elementary for well trained upper classmen. Perhaps this latter material could have been replaced by more of the genuine additional enrichment material which is scattered through the book particularly in Chapter 13, "Algebraic Operations and our Number System" where, for example, Fermat's theorem and Euler's ϕ function are used in a discussion of repeating decimals.

Weaknesses of the book lie in the omission of adequate discussions of several significant trends and emphases in mathematical education such as: the teaching of geometry without a text, stressing its structure and the nature of proof, and the use of teaching aids, field work, and laboratory methods. Further, intuitive geometry receives little attention, problem solving as a general objective is not adequately discussed and the work of the Post War Plans Committee of the National Council of Teachers of Mathematics is ignored (except for a brief mention of its *Guidance Pamphlet*), as are several other significant committee reports.

One suspects that these omissions were the result of combining the neces-

sity for limiting the text with the author's preferences in mathematical educational philosophy. However, teachers should be aware of and evaluate conflicting views, different procedures and important reports. The author's own classroom program from whence the book stems seems also to be reflected in the style which is occasionally telegraphic and occasionally repetitive as if he had written up at separate times the notes for a number of units without fully surveying the whole to eliminate unnecessary repetition and to correlate the parts. For example, section 15-6 "Professional Activities" in the chapter on *Professional Duties and Teacher Preparation* is almost identical with section 16-9 "Professional Growth of Teachers" in the chapter on *Supervision and Improvement of Instruction*.

The book is remarkable for a much more complete discussion of the foundations of mathematics than is customarily found in such a text as well as for its discussion of junior college and advanced high school courses. However, granting that complete rigor is not desirable on an elementary level, the reviewer would have liked to see more care displayed in the definitions of a *function*, *continuity*, *locus*, and *conditional equation*. The postulates for algebra on page 129 are far from complete but no reference is made either to this fact or to a source for a complete set. Further, the discussion of approximate computation ignores completely the difference between decimal places and significant figures and between the rules for addition and multiplication of approximate numbers. The "odd-add" rule is treated well as is scientific notation.

The author's repeated insistence on teaching for meaning (understanding of basic principles) leaves one surprised that he advocates the use of the term and process "transposition" (after it has been rationalized) and that he uses the ":" symbolism for ratios in secondary school geometry although he stresses teaching proportion from the algebraic equation viewpoint. These were frowned upon as far back as the 1923 report of the Joint Committee on the *Reorganization of Mathematics in Secondary Education*, an important document to which the author does refer several times.

Only a few minor typographical errors were noted. On page 291 the cosine law is named where the sine law is intended. The format is pleasing, the headings and outlines are helpful, but the shiny paper makes the book hard to read in some lights.

It is a large task to put into one book all that the author proposes. It requires selection and condensation. The author has done this on the avowed basis of his experience in teaching both mathematics and mathematics instructors, and implicitly on the basis of his personal beliefs and tastes in regard to desirable emphases in mathematics teaching. Not everyone will agree with the omissions, but few will debate the main themes and emphases of the book.

In general, it is a book to parts of which the reviewer will refer his students and is one of the few usable texts now in print in this field.

P. S. Jones
University of Michigan

NEW BOOKS RECEIVED

Mathematics of Investment. By P. R. Rider and C. H. Fischer. New York, Rinehart and Company, 1951. 12+359 pages. $5.00.

Finite Matrices. By W. L. Ferrar. New York, Oxford University Press, 1951. vi+182 pages. $4.00.

Tensor Analysis, Theory and Applications. By I. S. Sokolnikoff. New York, John Wiley and Sons, Inc., 1951. ix+335 pages. $6.00.

Spinoza Dictionary. By R. D. Dagobert. New York, Philosophical Library, Inc., 1951. xiv+309 pages. $5.00.

Advanced Engineering Mathematics. By C. R. Wylie, Jr. New York, McGraw-Hill Book Company, 1951. xiii+640 pages. $7.50.

Philosophical Problems of Mathematics. By Baron v. Freytag gen Loringhoff Bruno. New York, Philosophical Library, 1951. 88 pages. $2.75.

Intermediate Algebra for Colleges. By J. B. Rosenbach and E. A. Whitman. New York, Ginn and Company, 1951. x+219+xxii pages+a four-place common logarithm table of numbers (in pocket on back cover). $3.00.

Calculus. Revised Edition. By J. V. McKelvey. New York, The Macmillan Company, 1951. 7+405 pages. $4.50.

Mathematics for Technical Students, Book III. By J. D. N. Gasson. Cambridge, The University Press, 1951. xii+451 pages. $3.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to H. D. Larsen, Albion College, Albion, Michigan.

Pi Mu Epsilon, Iowa State College

During the school year 1950–51, the following papers were presented at meetings of the *Iowa Alpha* chapter of *Pi Mu Epsilon*:

Paradoxes simple and complex, by Dr. Joe Foote

The use of matrix notation for transfer functions of physical systems, by Mr. A. J. Bradt

Some equations for adsorption of gases, by Dr. Robert S. Hansen.

Eight meetings were held during the year, including two initiation banquets in the fall and spring. The Pi Mu Epsilon prize, which is given each year to an outstanding mathematics student, was awarded to Robert H. Tweedy.

Officers elected for 1951–52 are: Director, Richard E. Johnson; Vice-Director, John Druyor; Secretary, Caroline Iverson; Treasurer, Donald Klippenstein; Faculty Advisor, Ralph M. Robinson.

Mathematics Club, Iowa State College

The *Iowa State College Mathematics Club* is sponsored by the Iowa Alpha chapter of Pi Mu Epsilon, the fraternity Vice-Director acting as student-director of the club. Richard E. Johnson is the retiring and John Druyor is the new student-director. Three meetings were held during 1950–51 at which the following programs were presented:

Statistics: what it is and what it can do, by Dr. Bernard Ostle

Movie: The Pythagorean theorem

Prizes in math competition, by Miss Lieberknecht.

Mathematics Club, Los Angeles City College

The *Los Angeles City College Mathematics Club*, which was reorganized in September of 1950 after a lapse of several years, held bi-weekly meetings during 1950–51. City College staff members delivered the following lectures:

Continued fractions, by Mr. Samuel Skolnik

Rabbits and rectangles: Fibonacci series, by Mr. Benjamin K. Gold

The four-color problem, by Mr. Robert B. Herrera

The theory of relativity, by Mr. Samuel Skolnik

Napier's logarithms, by Mr. Fred Marer

Electronic computing machines, by Mr. George Mapes.

The following lectures by visiting speakers were presented for the students and faculties of both Los Angeles City College and Los Angeles State College. The first, third and fifth lectures were sponsored by L. A. State, the second and fourth lectures by L. A. City:

Foundations of mathematics, by Professor Hans Reichenbach

Pythagorean number triples, by Dr. Tobias Dantzig

Interplanetary navigation, by Professor Robert Herrick

Mathematical logic, by Professor Leon M. Henkin

Properties of zero and infinity, by Professor Samuel E. Urner.

The officers for 1950–51 were: President, Leroy K. Barickman; Vice-President, Arthur Freed; Secretary-Treasurer, Henry P. Thompson; Faculty Sponsor, Robert B. Herrera.

Pi Mu Epsilon, University of Richmond

During 1950–51, the *Virginia Alpha* chapter of *Pi Mu Epsilon* held nine meetings at which the following papers were presented:

Mathematical side of astronomy, by Mr. Ogburn

Topology, by Miss Marion Jeffries

Opportunities in the field of mathematics, by Dr. C. H. Wheeler III, Samuel Smith, and Barbara McGeehee

A glimpse of time, by Miss Isabel Harris.

Officers elected for 1951–52 are: Director, Dr. C. H. Wheeler III; Vice-Director, Willard E. Meador, Jr.; Secretary, James L. Judson; Treasurer, William F. Herget.

Pi Mu Epsilon, Hunter College

During 1950–51, the following papers were presented at meetings of the *Hunter College* chapter of *Pi Mu Epsilon*:

Cerva's theorem and applications, by Hannah Stein

Menelaus' theorem and applications, by Ursel Kramer

Miquel theorem, pedal triangle, Simson line, by Marie Barbiere

Brocard configuration, by Dorothy Freudenberger

Euler's line, by Virginia Cacich

Nine-point circle, by Muriel Teitell

Inscribed and escribed circles of a triangle, by Stanley Gaffen

Theorem by Feuerbach, by Rita Goldstein

History of cartography, by Jane Luhks

The terrestrial sphere, by Norma Lackow

Equidistant azimuthal projection and UN flag, by Alice Gersh

Stereographic projection, by Mildred Spiewak

Equations of transformation by cartesian coordinates; *applications of theory of inversion to analytic geometry*, by Evelyn Horvath

Geometric construction of inverse points by compasses alone; *applications of theory of inversion to modern geometry*, by Margot Henlein

Gerardus Mercator and Mercator projection, by Elaine Gross

The celestial sphere, by Mildred Spiewak.

Professor Gustav A. Hedlund of Yale University was guest speaker at the May initiation; his topic was *Clifford Klein space forms or small geometries.*

Officers for the second semester were: Director, Prof. Laura Guggenbuhl; President, Mildred Spiewak; Corresponding Secretary, Norma Lackow; Recording Secretary, Rita Goldstein; Treasurer, Margot Henlein.

Ricci Mathematics Academy, Boston College

The *Ricci Mathematics Academy of Boston College* held regular semi-monthly meetings during 1950–51 at which the following papers were presented:

The applications of visual aids in mathematics, by Professor H. W. Syer

Introduction to plane geometry, by Mr. E. J. Montella

The duties and responsibilities of the actuary, by Mr. W. L. O'Connor

Electronic digital computers, by Mr. N. Cedrone

Engineering work and sales liaison, by Mr. P. J. Equi.

Also presented were films on algebra, geometry, navigation, and nautical astronomy. The *Ricci Mathematical Journal*, official publication of the Academy, appeared in November, February, and April.

The spring meeting of the Greater Boston Intercollegiate Mathematics Clubs Association was held on April 26 at Boston College, at which Prof. Hans G. Haefeli presented a paper on *The use and development of geometry.*

Officers of the Academy were: President, Charles W. Hunter; Vice-President, William J. Kelly; Secretary, Charles L. Niles, Jr.; Treasurer, William J. Broderick.

Mathematics Club, Brown University

The thirty-seventh annual program of the *Mathematics Club of Brown University* consisted of six regular meetings and the club picnic. The papers presented at the meetings in 1950–51 were:

Models of vibrating systems, by Professor Erastus Henry Lee
Cubic equations, by James Gray
Boolean algebra, by Simone Matteodo
Women versus mathematics, by Dean Nancy Duke Lewis, Pembroke College
What is wrong with this picture? by Dora Bucco
Planetary motion, by Paul Duclos
Generalising the factorial, by William Youden
How big is the factorial? by Barbara Webb
Theory of games, by Professor David Gale.

The Committee on Program and Arrangements was: Chairman, Simone Matteodo; Secretary, Barbara Webb; Treasurer, James Gray; Faculty Advisor, Professor Frank Stewart; and Martin Badoian, Edwin Boynton, Louise O'Donnell, Patricia Wandelt, Robert Warren, William Youden.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Assistant Professor P. B. Burcham of the University of Missouri represented the Association at the inauguration of President M. E. Collins of Missouri Valley College on December 10, 1951.

Boston University announces: Dr. J. B. Giever and Dr. F. J. Scheid have been promoted to assistant professorships; Dr. G. E. Noether of New York University has been appointed Assistant Professor of Mathematical Statistics; Dr. Felix Browder of Massachusetts Institute of Technology has been appointed to an instructorship; Assistant Professor Ottam Chand has returned to India; Instructor N. G. Blomquist has returned to the Institute of Mathematical Statistics, Stockholm, Sweden.

Georgia Institute of Technology reports the following: Instructor L. R. Daniel has been promoted to an assistant professorship; Mr. M. B. Sledd has been appointed Research Associate Professor; Dr. R. A. Willoughby has been appointed to an assistant professorship; Dr. W. B. Evans has been called to active service with the United States Air Force.

Louisiana State University announces the following: Assistant Professor E. S. Elyash of the University of Pittsburgh and Assistant Professor E. G.

Kundert of the University of Tennessee have been appointed to assistant professorships; Dr. Eugenio Calabi is on leave of absence for the academic year 1951–52 and is at Princeton University; Dr. H. C. Wang is also on leave for the year 1951–52 and is at the Institute for Advanced Study.

Massachusetts Institute of Technology makes the following announcements: Dr. G. F. D. Duff and Dr. J. F. Nash, Jr., have been appointed to C. L. E. Moore instructorships; Dr. D. E. Richmond has been appointed Lecturer.

Southern Methodist University announces that Mr. John Derr and Mrs. I. D. Russell have been appointed to instructorships.

St. Bonaventure University reports: Reverend M. V. Perry, formerly of Siena College, has been appointed to an instructorship; Reverend Joseph Ruther, formerly an instructor, is now at St. Joseph Seminary, Callicoon, New York; Assistant Professor Robert Smith and Instructors Donald Smith and Philip Wolf have accepted positions in industry.

At Texas Christian University: Assistant Professor L. A. Colquitt has been promoted to an associate professorship; Miss Margaret Welch has been appointed to a graduate fellowship.

The University of Alabama announces: Dr. O. R. Ainsworth, previously a graduate assistant at the University of California, has been appointed to an assistant professorship; Professor J. D. Mancill has been granted a leave of absence in order to accept a position as Mathematics Consultant at Redstone Arsenal, Huntsville, Alabama.

The University of California at Los Angeles makes the following announcements: Associate Professor Clifford Bell has been promoted to a professorship; Dr. R. M. Redheffer has been promoted to an assistant professorship; Dr. Edward Stiefel of Eidenossische Technische Hochschule, Zurich, Zwitzerland, has been appointed Visiting Professor; Dr. Enrique Cansado of the Bureau of the Budget, Washington, D. C., has been appointed Visiting Assistant Professor; Dr. Maria Weber, formerly Bateman research associate at California Institute of Technology, has been appointed Lecturer; Dr. Barrett O'Neill, previously a graduate student at California Institute of Technology, has been appointed to an instructorship; Professor G. E. F. Sherwood has retired with the title of Professor Emeritus.

The University of Connecticut announces that Dr. G. B. Robison and Dr. Henry Zatzkis have been appointed to instructorships and that Miss Ann M. Curran and Miss Cecilia T. Welna have been appointed to assistant instructorships.

At the University of Illinois: Associate Professor Harry Levy has been promoted to a professorship; Assistant Professor Herman Chernoff has been promoted to the position of Associate Professor of Mathematical Statistics; Assistant Professors L. L. Steimley and H. E. Vaughn have been promoted to associate professorships; Professor Beno Eckmann of Zurich, Switzerland, was Visiting Professor during the first semester of 1951–52; Professor W. J. Trjitzinsky is on sabbatical leave during the second semester; Associate Professor G. P.

Hochschild is on leave for the academic year and is at Yale University; Associate Professor Herman Chernoff and Assistant Professor M. E. Munroe were on leave during the first semester; Assistant Professor F. E. Hohn has received a Ford Foundation award for 1951–52.

The University of Kansas announces the following: Professor G. W. Smith has retired as Chairman of the Department of Mathematics but is continuing to teach in the Department; Professor G. B. Price has been appointed Chairman of the Department; Professors Nachman Aronszajn and A. H. Diamond of Oklahoma Agricultural and Mechanical College have been appointed to visiting professorships; Dr. S. S. Shrikhande of the College of Science, Nagpur, India, has been appointed Visiting Assistant Professor; Dr. W. K. Moore, previously a graduate student at the University of Kansas, and Mr. A. J. Zeichner, formerly an instructor at Oklahoma Agricultural and Mechanical College, have been appointed to instructorships.

At the University of Maryland: Dr. G. S. S. Ludford, previously Choate fellow at Harvard University, has been appointed to an assistant professorship; Mr. N. Z. Wolfsohn, formerly teaching and A. E. C. fellow at Harvard University, has been appointed to an assistant professorship; Mr. G. L. Spencer II, who has been a teaching and research assistant at the University of Michigan, has been appointed to an instructorship; Associate Professor J. L. Vanderslice is on leave of absence; Professor M. H. Martin has been appointed Chairman of the Committee on Applied Mathematics of the American Mathematical Society for a three year term beginning January 1, 1951 and Editor of the Proceedings of the Fourth Symposium on Applied Mathematics.

The University of Mississippi reports the following: Dr. L. L. Scott, previously a graduate student at the University of Illinois, has been appointed to an assistant professorship; Assistant Professor R. D. Sheffield is on leave of absence for the academic year 1951–52 for the purpose of graduate study at the University of Tennessee.

The University of Pennsylvania announces: Dr. H. E. Rauch, formerly of the Institute for Advanced Study, has been appointed Associate; Professor I. J. Schoenberg is on leave of absence and is at the Institute for Numerical Analysis.

The University of Rochester announces the following: Assistant Professor Dorothy L. Bernstein has been promoted to an associate professorship; Mr. Robert MacDowell, previously a graduate student and part-time instructor at the University of Michigan, has been appointed to an instructorship; Dr. Arthur Dutton, formerly an instructor and research assistant in Statistics at Iowa State College, has been appointed Assistant in Atomic Energy Project and Lecturer in Mathematical Statistics; Mr. W. A. Small, who has been a graduate student at the University, has been appointed to a part-time instructorship.

The University of Saskatchewan reports that Dr. James Sanders, previously a graduate student at Syracuse University, and Mr. Nathan Shklov, formerly a graduate student at the University of Indiana, have been appointed to instructorships.

At the University of Utah: Dr. Cecil Burgess, formerly a part-time instructor at the University of Texas, Dr. Henry Hiz of the Department of Philosophy of the University of Pennsylvania, and Dr. Eugene Parker, previously a graduate assistant at California Institute of Technology, have been appointed to instructorships; Professor I. O. Horsfall has retired.

Wayne University makes the following announcements: Associate Professor D. C. Morrow has been promoted to a professorship; Assistant Professor Arvid Jacobson has been promoted to an associate professorship; Instructor Samuel Conte has been promoted to an assistant professorship; Mr. George Johnson, who has been with the Kaiser-Frazer Corporation, Mr. Charles Kraft, a graduate student of the University of California at Berkeley, and Mr. Martin Wechsler, a graduate student at the University of Michigan, have been appointed to instructorships.

West Virginia University announces the following: Assistant Professor Margaret B. Cole has been promoted to an associate professorship; Mr. C.N. Cochran, Mr. F. E. Forsythe and Mr. P. S. Null have been appointed to instructorships; Professor C. N. Reynolds has retired with the title of Professor Emeritus.

Mr. L. G. Arnold, previously a part-time instructor at the University of Rochester, is now with the Navy Ordnance Division of Eastman Kodak Company.

Dr. Elsie T. Church has been appointed to an associate professorship at Northwestern State College of Louisiana.

Instructor Robert Coke of Southern Methodist University is now an instructor at Sheppard Field, Wichita Falls, Texas.

Associate Professor G. N. Conwell of the University of Georgia has retired.

Instructor J. A. Englund, formerly of Massachusetts Institute of Technology, has been appointed to an instructorship at Creighton University.

Associate Professor Howard Eves of Oregon State College has been appointed Professor and Head of the Department of Mathematics of Champlain College.

Instructor D. O. Gray, who has been promoted to an assistant professorship at the University of Houston, is on military leave with the United States Navy.

Assistant Professor G. P. Loweke of the College of Engineering of Wayne University has been promoted to an associate professorship.

Associate Professor Kenneth May of Carleton College has been promoted to the position of Head of the Department of Mathematics.

Professor W. T. Reid is on leave of absence from Northwestern University in order to serve as a staff member of the Sandia Corporation, Albuquerque, New Mexico.

Professor L. S. Shively of Ball State Teachers College, Indiana, has retired with the title of Professor Emeritus.

Associate Professor J. H. Simester of the Speed Scientific School, University of Louisville, has been promoted to a professorship.

Sister M. Tarcisius, formerly of Mt. Mercy and St. Mary's High School, Pittsburgh, Pennsylvania, is now at St. Peter's Convent, McKeesport, Pennsylvania.

Dr. M. M. Slotnick has been promoted to the position of Chief Mathematician, Humble Oil and Refining Company, Houston, Texas.

Professor L. W. Stark has accepted a position at Redstone Arsenal, Huntsville, Alabama.

Assistant Professor H. W. Syer of the School of Education, Boston University, has been promoted to an associate professorship.

Mr. C. M. Terry has a position as Analyst with the Department of Defense, Washington, D. C.

Lecturer F. B. Thompson of the University of California at Berkeley has accepted a position as Mathematician with the Rand Corporation, Santa Monica, California.

Mr. W. E. Timon, Jr., formerly a graduate fellow at Tulane University, is employed by the Esso Standard Oil Company Refinery, Baton Rouge, Louisiana, and has a part-time teaching position at Louisiana State University.

Mr. N. D. Vlachos, who has been an instructor at Drexel Institute of Technology, has a position as Mathematician at the Kellett Aircraft Corporation, Camden, New Jersey.

Instructor B. T. Wade of Kent State University has been appointed to an assistant professorship at Clemson Agricultural College.

Mr. O. L. Wadkins, Jr., formerly at White Sands Proving Ground, has a position as Research Engineer with North American Aviation, Los Angeles, California.

Mr. Alan Wayne, who has been teaching at Automotive High School, Brooklyn, has been appointed Chairman of the Department of Mathematics and Science, Williamsburg Vocational High School, Brooklyn, New York.

Mr. Edmund Wesolowski, previously a student at Elmhurst College, has a position as a spectrographer at Pure Oil Research and Development Laboratories, Crystal Lake, Illinois.

Assistant Professor R. H. Wilson of Temple University has been appointed to an assistant professorship at the University of Louisville.

Miss Laura Blakeley of Armstrong College died on October 19, 1951.

Assistant Professor Emeritus R. D. Daugherty of Kansas State College died on December 1, 1951.

Professor Emeritus H. L. Hall, retired head of the Department of Mathematics of Northwestern State College, Oklahoma, died on November 18, 1951.

Mr. J. F. Hubbard of J. H. Goddard and Company, Boston, Massachusetts, died on September 24, 1951.

Professor Emeritus N. J. Lennes of Montana State University died on November 21, 1951. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-FIFTH ANNUAL MEETING OF THE ASSOCIATION

The thirty-fifth annual meeting of the Mathematical Association of America was held at Brown University, Providence, Rhode Island, on Saturday, December 29, 1951, in conjunction with the annual meeting of the American Mathematical Society. A total of five hundred and sixty persons were registered, including the following two hundred and seventy-two members of the Association:

C. R. Adams, J. B. Adkins, R. P. Agnew, L. V. Ahlfors, M. I. Aissen, R. G. Albert, C. B. Allendoerfer, R. D. Anderson, R. C. Archibald, H. E. Arnold, L. G. Arnold, Max Astrachan, W. E. Barnes, Grace E. Bates, J. D. Baum, W. R. Baum, Helen P. Beard, Ralph Beatley, W. R. Beck, E. G. Begle, H. A. Bender, A. A. Bennett, H. G. Bergmann, Brother Bernard Alfred, S. D. Bernardi, Barbara B. Betts, Garrett Birkhoff, D. W. Blackett, Shirley A. Blackett, S. G. Bourne, Julia W. Bower, Robert Breusch, H. W. Brinkmann, Bailey Brown, H. D. Brunk, J. A. Bullard, R. S. Burington, R. A. Burrows, L. J. Burton, Jewell H. Bushey, E. A. Cameron, Mildred E. Carlen, L. Virginia Carlton, T. C. Carson, W. B. Carver, W. F. Cheney, Jr., Sarvadaman Chowla, G. R. Clements, R. N. Cobb, L. W. Cohen, Nancy Cole, E. M. Cook, A. H. Copeland, Richard Courant, R. R. Coveyou, Robert Davies, W. M. Davis, A. H. Diamond, H. L. Dorwart, M. P. Dossey, F. G. Dressel, G. R. Dube, Nelson Dunford, Ruth B. Eddy, Samuel Eilenberg, Benjamin Epstein, D. H. Erkiletian, Jr., A. B. Farnell, W. E. Ferguson, H. E. Fettis, F. A. Ficken, C. D. Firestone, J. R. Foote, L. R. Ford, Tomlinson Fort, J. S. Frame, H. D. Friedman, Orrin Frink, J. W. Gaddum, R. W. Gardner, A. E. Gault, H. M. Gehman, Irving Gerst, K. G. Getman, W. M. Gilbert, R. E. Gilman, R. D. Glauz, A. M. Gleason, H. E. Goheen, W. O. Gordon, W. H. Gottschalk, Arthur Grad, H. E. H. Greenleaf, J. B. Gregg, Florence N. Greville, T. N. E. Greville, Emil Grosswald, Seymour Haber, Franklin Haimo, P. C. Hammer, Frank Harary, B. T. Harris, W. L. Hart, Frank Hawthorne, G. A. Hedlund, M. H. Heins, Melvin Henriksen, Fritz Herzog, T. H. Hildebrandt, Einar Hille, W. M. Hirsch, S. P. Hoffman, D. L. Holl, L. Aileen Hostinsky, E. Marie Hove, Aughtum S. Howard, C. C. Hsiung, M. Gweneth Humphreys, W. A. Hurwitz, L. C. Hutchinson, R. N. Johanson, L. W. Johnson, R. E. Johnson, Roberta F. Johnson, B. W. Jones, F. B. Jones, O. C. Juelich, G. K. Kalisch, L. H. Kanter, Irving Kaplansky, J. H. B. Kemperman, D. E. Kibbey, H. S. Kieval, V. L. Klee, S. C. Kleene, J. R. Kline, Morris Kline, J. C. Knipp, R. L. Korgen, H. W. Kuhn, C. E. Langenhop, E. H. Larguier, J. A. Larrivee, R. D. Larsson, E. H. Lee, P. J. Leonard, Tadeusz Leser, C. J. Lewis, F. W. Light, Jr., Sally I. Lipsey, Lee Lorch, R. C. Lyndon, L. A. MacColl, Saunders MacLane, H. M. MacNeille, H. P. Manning, R. J. Marcou, Morris Marden, M. H. Martin, W. T. Martin, V. O. McBrien, N. H. McCoy, A. E. Meder, Jr., D. F. Mela, Karl Menger, W. E. Mientka, W. H. Mills, Deane Montgomery, T. W. Moore, R. K. Morley, A. J. Mortola, B. H. Mount, Jr., Pauline E. Mount, W. L. Murdock, Paul Nesbeda, C. V. Newsom, E. N. Nilson, C. O. Oakley, L. F. Ollmann, Oystein Ore, E. F. O'Shea, J. C. Oxtoby, D. K. Pease, W. O. Pennell, P. M. Pepper, I. E. Perlin, H. P. Pettit, C. R. Phelps, J. C. Polley, Walter Prenowitz, G. B. Price, M. H. Protter, Tibor Rado, J. F. Randolph, G. E. Raynor, C. J. Rees, Mina S. Rees, Eric Reissner, C. N. Reynolds, Harris Rice, D. E. Richmond, G. deB. Robinson, Selby Robinson, G. B. Robison, I. H. Rose, Alex Rosenberg, P. C. Rosenbloom, Arthur Rosenthal, Jane S. Rothe, H. L. Royden, Evelyn C. Rusk, Charles Saltzer, Arthur Sard, J. A. Schatz, M. A. Scheier, Edith R. Schneckenburger, I. J. Schoenberg, Augusta L. Schurrer, Seymour Schuster, Abraham Schwartz, B. L. Schwartz, C. H. W. Sedgewick, Esther Seiden, Sister Mary Felice, Sister Mary Petronia, E. C. Smith, Jr., Robert Edward Smith, Andrew Sobczyk, Elizabeth S. Sokolnikoff, Joseph Spear, Marion E. Stark, E. P. Starke, J. R. K. Stauffer, F. M. Stewart, Ruth W. Stokes, J. S. Stubbe, R. L. Swain, Otto Szasz, Gabor Szego, R. C. Taliaferro, William Clare Taylor, G. B. Thomas, Jr., J. M. Thomas, D. L. Thomsen, R. M. Thrall, W. J.

Thron, John Todd, Marian M. Torrey, J. I. Tracey, A. W. Tucker, E. P. Vance, Helen E. Van Sant, C. H. Vehse, John von Neumann, T. L. Wade, Jr., R. W. Wagner, S. E. Walkley, William Wallace, J. L. Walsh, Mother Mary E. Walsh, W. H. Warner, S. E. Warschawski, J. V. Wehausen, Marie J. Weiss, P. M. Whitman, G. T. Whyburn, D. V. Widder, W. L. Williams, D. M. Young, Jr., Oscar Zariski, J. L. Zemmer, Jr., J. A. Zilber.

Sessions of the Association were held on Saturday morning and afternoon in the Auditorium of the Metcalf Chemical Laboratory, Brown University. President Saunders MacLane presided at both sessions. The Program Committee for the meeting consisted of N. H. McCoy, Chairman, Bailey Brown, and Moses Richardson.

FIRST SESSION OF THE ASSOCIATION

"Partially Ordered Sets," by Professor Orrin Frink, Jr., Pennsylvania State College.

"Trends in Algebraic Topology," by Professor N. E. Steenrod, Princeton University.

"Cardano's Book on Gambling and Probability," by Professor Oystein Ore, Yale University.

SECOND SESSION OF THE ASSOCIATION

Symposium on the Teaching of Calculus

"Variables in Calculus," by Professor Karl Menger, Illinois Institute of Technology.

"Insight and Understanding in the Calculus," by Professor Walter Prenowitz, Brooklyn College.

"Concepts Versus Rigor," by Professor Herbert Federer, Brown University.

"Notion and Definition of a Function," by Professor J. F. Randolph, University of Rochester.

"Combining Analytic Geometry with Calculus," by Professor A. W. Tucker, Princeton University.

MEETING OF THE BOARD OF GOVERNORS

The Board met on Friday morning in the Faunce Memorial Room of Faunce House. Nineteen members of the Board were present. Among the more important items of business transacted were the following:

The Board voted to approve the appointment by President MacLane of the following Nominating Committee for 1952: W. L. Duren, Chairman, E. P. Northrop, and R. J. Walker.

To fill the vacancy in the Board caused by the death of Professor J. B. Rosenbach, the Board voted to elect Professor F. H. Steen of Allegheny College as Governor from the Allegheny Mountain Section for a term expiring on June 30, 1954.

To succeed Dr. C. V. Newsom as a representative of the Association on the

Policy Committee for Mathematics, the Board elected Professor J. S. Frame of Michigan State College.

On the recommendation of the Executive Committee, the Board voted to re-elect Professor H. M. Gehman of the University of Buffalo as Secretary-Treasurer of the Association for a second term of five years, beginning on January 1, 1953.

The Board voted to approve the recommendation of the Committee on Expository Lectures that Professor Tibor Rado be invited to deliver a series of three expository lectures at the 1952 Summer Meeting.

On the recommendation of the Committee to Plan a Symposium on Teacher Education in Mathematics, the Board voted (subject to parallel action by the National Council of Teachers of Mathematics) that the Committee be authorized to determine the general plan and place of meeting for the Symposium and to appoint a Director. The Director will, by conference with the Committee, select consultants and put the plan for the Symposium into operation.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Saturday, December 29, 1951, at 2:00 P.M. in the Auditorium of the Metcalf Chemical Laboratory of Brown University, Providence, Rhode Island.

The Secretary announced the results of the balloting for officers, in which 1152 votes were cast. F. L. Griffin of Reed College was elected First Vice-President for the two-year term, 1952–1953. D. H. Lehmer of the National Bureau of Standards, Los Angeles, California and W. E. Milne of Oregon State College were elected Governors for the three-year term, 1952–1954.

The Secretary read the following resolution adopted by the Board of Governors:

> "The Board of Governors wish to express their deep sense of appreciation to Dr. Carroll V. Newsom for his long service as an editor of the American Mathematical Monthly. Beginning in 1943, when he started a widely useful section on War Information, and ending at the present time after a five year span as Editor-in-Chief, he has given freely of his time and his talents. We commend him for his steady devotion and his fine sense of duty."

MEETINGS OF OTHER ORGANIZATIONS

The sessions of the American Mathematical Society began on Wednesday, December 26, and continued through Friday afternoon. The Josiah Willard Gibbs Lecture was delivered by Dr. Kurt Gödel of the Institute for Advanced Study. The title of his address was "Some Basic Theorems on the Foundations of Mathematics and Their Philosophical Implications." Invited addresses were given by Professor Shizuo Kakutani of Yale University on "Brownian Motion" and by Professor G. P. Hochschild of the University of Illinois and Yale University on "Cohomology in Algebraic Number Fields."

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of F. M. Stewart, Chairman, C. R. Adams, A. A. Bennett, L. W. Cohen, H. M. Gehman, and E. H. Lee.

Registration headquarters was in the west wing of Faunce House on Waterman Street. Rooms in the University dormitories were available from Wednesday morning to Sunday morning. Meals were served in the University Refectory.

The ladies of the Mathematics Department, of the Graduate Division of Applied Mathematics and of the Department of the History of Mathematics entertained at tea in the Art Gallery of Faunce House on Wednesday afternoon.

The movie, "Ruggles of Red Gap," was shown in the Faunce House Theatre on Thursday evening.

On Thursday afternoon there was held a tour of Pendleton House, an example of an eighteenth-century Providence house. Tours through other similar houses were also held on Friday afternoon.

A number of special collections and exhibits were shown in the libraries of Brown University. These included exhibits of early mathematical books including the first printed edition of Euclid's "Elements."

The new offices of the American Mathematical Society and of Mathematical Reviews at 80 Waterman Street were open for inspection by those attending the meetings.

A joint dinner for the members of the mathematical organizations and their guests was held at 6:30 P.M. on Friday in the University Refectory. Professor C. R. Adams of Brown University acted as toastmaster. President John von Neumann of the American Mathematical Society extended the greetings of the Society. Mrs. Jewell H. Bushey, Second Vice-President of the Association, spoke about the past and present status of women mathematicians, stating certain theorems on this subject. President Henry M. Wriston of Brown University expressed the gratitude of the University at being able to entertain the mathematicians at this meeting.

Professor C. B. Allendoerfer presented a motion, which was adopted by a rising vote, expressing the deep appreciation of the members of the mathematical organizations for the hospitality which had been tendered to them at this meeting by Brown University. Especial thanks are due to the administrative staff, to the mathematicians of Brown University and to the local members of the Committee on Arrangements for their painstaking efforts in caring for the wants of those attending the meetings.

H. M. GEHMAN
Secretary-Treasurer

THE APRIL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the United States Naval Academy at Annapolis, Maryland, on Saturday, April 28, 1951. The Chairman of the Section, Professor O. J. Ramler of Catholic University, presided at both the morning and afternoon sessions.

A total of one hundred three persons registered their attendance at the meeting including the following sixty-nine members of the Association:

J. C. Abbott, R. P. Bailey, N. H. Ball, W. E. Barnes, T. J. Benac, B. H. Buikstra, L. E. Carville, Randolph Church, C. R. Clark, G. R. Clements, A. E. Currier, R. L. P. Eiseman, F. D. Faulkner, C. H. Frick, B. C. Getchell, R. A. Good, J. R. Gorman, E. S. Grable, E. C. Gras, Donald Greenspan, R. E. Greenwood, T. N. E. Greville, D. W. Hall, J. R. Hammond, S. B. Jackson, Walter Jennings, L. M. Kells, H. L. Kinsolving, A. E. Landry, W. W. Leutert, B. J. Lockhart, G. A. Lyle, C. G. Maple, K. F. McLaughlin, D. St. C. Melvin, Joseph Milkman, J. F. Milos, T. W. Moore, Paul Nesbeda, N. O. Niles, W. H. Norris, J. F. Paydon, A. J. Pejsa, J. W. Popow, F. M. Pulliam, O. J. Ramler, V. J. Rathbun, C. H. Rawlins, R. W. Rector, J. N. Rice, R. E. Root, C. A. Rupp, J. B. Scarborough, Paul Shapiro, W. F. Shenton, E. R. Sleight, Robert Edward Smith, C. A. Spicer, M. F. Stilwell, H. C. Stotz, E. G. Swafford, O. M. Thomas, J. A. Tierney, C. C. Torrance, Marian M. Torrey, John Tyler, J. H. White, P. M. Whitman, R. H. Wilson.

At the business meeting in the afternoon session the following officers were elected for the coming year: Chairman, Professor E. R. Sleight of the University of Richmond; Secretary, Professor C. H. Frick, Mary Washington College; Members of the Executive Committee, Professor Florence Mears, George Washington University, and Professor V. L. Klee, University of Virginia. In connection with the celebration of its fiftieth anniversary during 1951, the National Bureau of Standards has invited the section to hold its next meeting there. This invitation has been accepted, and the next meeting will therefore be held at the Bureau of Standards in Washington on December 8, 1951.

The morning session was scheduled to consist of five contributed papers as follows, though the fifth was actually postponed to the afternoon session in order that lunch could be served at 1:00 P.M.

1. *The method of enclosures for nonlinear differential equations*, by Professor A. R. Craw, United States Naval Academy, introduced by Professor R. P. Bailey.

The Poincaré-Bendixson Theorem was used to establish sufficient conditions for the existence of periodic solutions of non-linear differential equations. A demonstration was then given of the existence of periodic solutions of the van der Pol equation for certain values of the parameter.

2. *Some curves associated with laws of growth*, by Professor C. H. Frick, Mary Washington College.

If a quantity y is given as a function of some parameter t, the equation $y^* = y(t+1)$, the corresponding curve $y^* = F(y)$ in the y, y^*-plane is a curve related to this law of growth. Since the situation $y^* = y$ indicates the limiting situation or state of full growth, the intersections of the curve with the line $y^* = y$ are of particular interest. It was shown in a number of examples how the geometry of this curve is related to finding such limits. The method gives some results for summing divergent series.

3. *Napier and his logarithms*, by Miss Stirling Clark, University of Richmond, introduced by Professor E. R. Sleight.

This paper contained a brief history of the development of mathematics before Napier's time, together with the educational qualifications of this great Scotsman. Napier's rods as an aid to multiplication were discussed. Also the method was explained which was used in formation of logarithmic tables, Napier's best known accomplishment.

4. *Closure and complementation as operators*, by Mr. R. L. Eiseman, University of Maryland.

It has been shown by Donal Adelman that the operators of closure and complementation, applied successively to any set in a space satisfying Hausdorff's first three axioms, can produce at most fourteen distinct sets. R. A. Good discovered a set in the euclidean plane which actually yields all fourteen sets. It is possible to find sets which produce exactly each even number of sets from two to fourteen. The null set in the null space produces exactly one. There are no sets which will produce exactly any odd number of sets greater than one.

5. *An application of elliptic functions to plane flow*, by Professor J. O. Tierney, United States Naval Academy.

A steady, irrotational, sentropic plane flow governed by Frankl's equation $\psi_{\sigma\sigma}+k(\sigma)\psi_{\theta\theta}=0$ is determined by the choice of the function $k(\sigma)$. With suitable units, for a polytropic gas it is found that $k(\sigma)=\sigma-\frac{1}{2}\delta^2\sigma^2+\cdots$. Employing σ and the stream function ψ as independent variables, M. H. Martin has shown that Frankl's equation is equivalent to the equation.

$$\theta_{\sigma\sigma}=\left(\frac{K(\sigma)+\theta_\sigma^2}{\theta_\psi}\right)_\psi. \tag{1}$$

By taking $K(\sigma)=\sigma$ as a first approximation, Frankl's equation reduces to Tricomi's equation which has been used by various authors in studying transonic flow.

To obtain a better approximation, consider $K(\sigma)=\sigma-\delta^2\sigma^2/2$. By assuming θ as a quadratic function of σ, substitution in (1) and use of the method of undetermined coefficients leads to a system of ordinary differential equations and thence to the following solution of (1).

$$\theta=(2\delta^2)^{-1}\{\sqrt{2}\ln\left[2^{-1/2}\,\mathrm{nc}\,\delta\psi(\mathrm{sn}\,\delta\psi+\sqrt{2}\,\mathrm{dn}\,\delta\psi\right]-\mathrm{sn}\,\delta\psi\,\mathrm{dn}\,\delta\psi(1-\delta^2\psi)^2\}.$$

This is a generalization of the solution $\theta=\sigma\psi+\psi^3/3$ studied by Martin for the case $K(\sigma)=\sigma$.

6. *On truncation error and stability* by Professor W. W. Leutert, University of Maryland.

This was an invited address. If a partial differential equation is replaced by a partial difference equation, the truncation error is defined as the difference between the exact solution u of the boundary or initial value problem for the differential equation and an exact solution v of a corresponding boundary or initial value problem for the difference equation.

Two simple examples, heat conduction in a slab and the vibrating string, were considered and it was shown that for any positive value of the mesh ratio $r=\Delta t/\Delta x^2$ and $r=\Delta t/\Delta x$ respectively, there exist infinitely many exact solutions of an initial value problem for the difference equation which converge to the exact solution of the differential equation as the size of the mesh is decreased.

If the mesh ratio r exceeds $\frac{1}{2}$ or 1 respectively, random round off errors in a numerical computation will in general increase indefinitely, i.e. the numerical process is no longer stable.

These results contradict statements in a recent paper by O'Brien, Hyman, Kaplan, *A Study of the Numerical Solution of Partial Differential Equations*, J. Math. Phys. 29, pp. 223–251 (1951),

especially the assertion on page 228 that for the second problem and $r>1$ convergence is "clearly impossible" and that "$r \leqq 1$ is a necessary condition for convergence."

S. B. JACKSON, *Secretary*

THE APRIL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at the College of St. Benedict in St. Joseph, Minnesota, on Saturday, April 28, 1951. Sessions were held in the forenoon, at luncheon and in the afternoon. Professors Kenneth May, W. R. McEwen and J. M. H. Olmsted (Chairman of the Section) presided at the respective sessions.

Sixty-seven persons attended the meeting, including the following forty-three members of the Association:

H. M. Anderson, F. J. Arena, H. S. Berg, C. R. Bonnell, K. H. Bracewell, R. W. Brink, L. E. Bush, W. H. Bussey, R. H. Cameron, E. J. Camp, Elizabeth Carlson, W. B. Fulks, C. B. Germain, Gladys Gibbens, H. W. Godderz, Ruby M. Grimes, J. E. Hafstrom, K. L. Hankerson, A. G. Hill, Hildegarde Horeni, G. H. Jaeger, E. R. Johnston, Reverend W. C. Kalinowski, Karlis Kaufmanis, P. G. Kirmser, Fulton Koehler, W. S. Loud, Kenneth May, W. H. McBride, W. R. McEwen, Sigurd Mundhjeld, W. D. Monro, M. J. Norris, J. M. H. Olmsted, P. A. Rognlie, L. W. Sheridan, Sister M. Joanne, F. C. Smith, A. G. Swanson, P. Y. Tani, F. J. Taylor, Takashi Terami, J. W. Wegner.

At the business meeting the following officers were elected for the coming year: Chairman, H. D. Colson, Bemidji State Teachers College; Secretary, F. C. Smith, College of St. Thomas; Executive Committee, C. S. Carlson, St. Olaf College, A. G. Hill, North Dakota State College, J. M. H. Olmsted, University of Minnesota.

By invitation of the Executive Committee, Professor M. D. Donsker of the University of Minnesota delivered an address at the morning session entitled *Probability Limit Theorems*.

The following short papers were presented:

1. *A Note on special solutions of partial differential equations related to Laplace's equation*, Professor O. E. Stanaitis, St. Olaf College, introduced by Professor C. S. Carlson.

Some polynomial solutions of the partial differential equation

$$\frac{\partial^2 w}{\partial \alpha^2} - \frac{\partial^2 w}{\partial \beta^2} - k^2 n(n+1)(\mathrm{sn}^2 \alpha - \mathrm{sn}^2 \beta)w = 0$$

have been established by E. T. Whittaker. In a different approach it is shown here that there exists a complete linearly independent system of polynomial solutions which is easily obtained explicitly by simple transformation. The paper concludes with remarks on the applicability of the method to related differential equations.

2. *Some relations among the roots of cubic equations*, Professor W. S. Loud, University of Minnesota.

If the roots of a cubic equation, r_1, r_2, and r_3 are distinct, there is a linear fractional transformation $f(z)$ which takes r_1, r_2, and r_3 into r_2, r_3, and r_1 respectively. The transformation $f(z)$ can be found directly from the cubic equation without calculation of the roots. If the equation

is $x^3+ax+b=0$, $f(z)=(Az+B)/(Cz+D)$, where $A=9b+(-4a^3-27b^3)^{1/2}$, $B=-2a^2$, $C=-6a$, and $D=-9b+(-4a^3-27b^3)^{1/2}$. If the coefficients of the cubic are rational, the coefficients of $f(z)$ are all rational if and only if the discriminant is a square. In this latter case when in addition the three roots are all irrational, $f(z)$ is the only linear fractional transformation among the roots with rational coefficients. In the case of distinct irrational roots, and rational coefficients in $f(z)$, with A, B, C, and D relatively prime integers, if it also happens that $AD-BC=\pm 1$, it is also true that the continued fraction expansions of the three roots are identical after finitely many terms. The equation $x^3-3x+1=0$, for which $f(z)=(z-1)/z$, has all the above special properties.

3. *Vibration of a string with periodic tension*, Mr. H. W. Godderz, University of Minnesota.

The problem of the vibrating string with periodic tension was discussed. The equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}(1 - 2\gamma \cos 2\omega t)\frac{\partial^2 y}{\partial x^2}$$

was transformed into Mathiew's equation of canonical form

$$\frac{d^2 f}{dz^2} + (a - 2q \cos 2z)f = 0.$$

For periodic, and non-periodic oscillatory bounded solutions there are definite restrictions placed on a which is a complicated function of q. The results of three special cases having stable solutions were given.

4. *Derivation by rotation of the distance from a line to a point*, Professor Kenneth May, Carleton College.

The familiar proof of the formula for the distance from a line to a point involves drawing an auxiliary line and consideration of two possible cases for its location. These complications are avoided by using the following proof based on rotation of the axes. Let the line and point be given by $x \cos \alpha + y \sin \alpha - p = 0$ and (x_0, y_0). Rotate the axes through the angle α. The line and point are now given by $x' = p > 0$ and (x_0', y_0'). The directed distance from the line to the point is in all cases $d = x_0' - p$. But $x_0' = x_0 \cos \alpha + y_0 \sin \alpha$, and the desired result follows. Since trigonometry must precede study of the normal form, rotations can easily be introduced before or at this point. This is only one of many simplifications which result from the full exploitation of transformations.

5. *A new mnemonic for* π *(mod 10)*, Professor F. J. Arena, North Dakota State College.

Most mnemonics give the value of π to about 31 or 32 digits. The thirty-third digit in the value of π is zero. The reason why mnemonics do not represent π beyond 32 digits may be because mathematicians do not know how to get past this zero. The following mnemonic has been written in order to show how we can get around this difficulty. By counting the number of letters in each word and reducing that number modulus 10, the following mnemonic gives the-value of π to more than 33 digits:

See, I have a great invention
To assist eager geometricians
Whose problems implicate circle's periphery
And if you required that number pi
Useful transcendental and distinguished constant
For to readily ascertain
Count successive hieroglyphic lettered ciphered word.

$\therefore \quad \pi = 3.14159\ \ 26535\ \ 89793\ \ 23846\ \ 26433\ \ 83279\ \ 50288\ \ 4\cdots.$

6. *A note on functions harmonic in the unit circle*, Professor W. B. Fulks, University of Minnesota.

The author presents a proof, believed to be new, of a result which is contained in more general theorems in the literature. However, the theorem presented seems not to have been explicitly stated. It is as follows: If $u(r, \alpha)$ is representable in the unit circle by a Poisson-Stieltjes integral and has radial boundary values everywhere zero, then $u(r, \alpha)$ is identically zero in the unit circle.

7. *The inversion of Brillouin zones for thermodynamic solutions of solids*, Mr. C. R. Bonnell, Minneapolis-Honeywell Regulator Company.

The paper demonstrated a method of translating Brillouin zones into reciprocal lattice unit cells by cyclic changes of subscripts of the associated wave vector for an oscillating lattice. The translation was shown to be necessary for the calculation of the normal modes of vibration of the system. Some simple rules were given to predict whether or not the partition function was invariant to the translation.

The general equation of state was derived and the translation carried out for the square lattice, the cubic lattice, the body-centered lattice, and the face-centered lattice.

8. *Nicollet was a mathematician*, Professor W. H. Bussey, University of Minnesota.

The name of Joseph N. Nicollet is well known in Minnesota. His name has been given to Nicollet Avenue and Nicollet Island in Minneapolis, to a county in Southern Minnesota, and to the "Infant Mississippi" which flows into Lake Itasca. Nicollet came to Fort Snelling, Minnesota, in 1836 primarily for the purpose of exploring the upper Mississippi Valley. He completed with scientific accuracy the work of Henry R. Schoolcraft who had recently named Lake Itasca as the "true source" of the Mississippi River.

What is not well known in Minnesota is that Nicollet, before he came to the U. S. A., was an accomplished mathematician, astronomer, and educator, and that his mathematical interests and accomplishments were in such diverse fields as mathematical astronomy and the mathematics of life insurance and probability.

Mr. Bussey's paper told about Nicollet's life in France and how it happened that, at the age of 42 years, he gave up a very successful and distinguished career there for an altogether different career in this country.

FRANKLIN C. SMITH, *Secretary*

THE APRIL MEETING OF THE KENTUCKY SECTION

The annual meeting of the Kentucky Section of the Mathematical Association of America was held at Eastern Kentucky State College, Richmond, Kentucky, on Saturday, April 28, 1951. Dr. R. S. Park, Chairman of the Section, presided at the morning and afternoon sessions.

Fifty-one members were present at the meeting, including the following twenty-nine members of the Association:

H. H. Berry, M. C. Brown, R. C. Brown, Jr., Elsie T. Church, Esther A. Compton, V. F. Cowling, H. H. Downing, Clarence Ford, A. E. Foster, A. W. Goodman, Charles Hatfield, Sr., Aughtum S. Howard, Tadeusz Leser, A. G. McGlasson, W. L. Moore, R. S. Park, Sallie E. Pence, D. W. Pugsley, Sara L. Ripy, G. G. Roberts, W. J. Robinson, W. C. Royster, W. S. Snyder, D. E. South, A. L. Stamper, Guy Stevenson, J. T. Vallandingham, J. A. Ward, and W. M. Zaring.

Officers elected at the meeting were: Chairman, J. A. Ward, University of Kentucky; Secretary, G. G. Roberts, Berea College. The annual meeting in

1952 is to be held at the University of Kentucky, Lexington, Kentucky.

The following papers were presented:

1. *Three dimensional representation of complex roots*, by Professor J. A. Ward, University of Kentucky.

Denote three mutually perpendicular axes in euclidean space by X, Y_r, and Y_i. Let $(a, b+ci)$, with a, b, and c real and $i^2=-1$, satisfy $f(x, y)=0$. The set of points (a, b, c) are called the "real-complex" graph of $f(x, y)=0$. The graphs of plane analytic geometry are identified with that portion of the "real-complex" graph which lies in the X Y_r-plane. Mr. Ward discussed these graphs for several elementary functions and pointed out that for a conic section the real-complex graph will always lie in two perpendicular planes.

2. *On a joint life probability*, by Mr. Sherman Vanaman, Jr., University of Kentucky, introduced by the Secretary.

Standard probability notation with certain extensions was used to prove the following joint life probability theorem: The probability that the joint existence of the lives (x) and (y) will fail in the $(n+1)$st year is given by

$$(A_n \cdot B_n) - (A_{n+1} \cdot B_{n+1})$$

where the symbol $(A_n \cdot B_n)$ represents the probability that (x) and (y) will both survive n years. The result was then used in developing the expression

$$A_{xy} = va_{xy} - a_{xy}$$

for the present value of a \$1 joint whole life insurance.

3. *A study of word frequencies*, by Mr. Strawn Taylor, University of Louisville, introduced by Professor Guy Stevenson.

This study is essentially a pilot study regarding large populations which are subdivided into many subsets, the written language. The objectives of the paper are to obtain answers to the following questions:

1. Can sampling be used to determine frequencies of occurrence of written words in a given population?
2. Does the given author use approximately the same frequency of words in his various publications?
3. Can patterns be established for a writer by which he may be identified?

In answer to the first question, it was discovered that a sample of at least 10,000 words is necessary for the inspection of the more frequently occurring words such as *the, of, and, on, etc.* Secondly, the data indicate that an author (Joyce) does not use the same frequency of words in his two books (*Ulysses* and *Finnegan's Wake*). Thirdly, Mr. Taylor found it possible to set up patterns for different authors. These patterns indicate quite clearly that two different authors (Joyce and Emerson in this case) vary distinctly in their use of words. Mr. Taylor emphasized that this is only a pilot study and that there is great need for much more counting and investigating of the English language, in order to establish all-inclusive norms.

4. *The Burkill integral as a unifying concept in undergraduate mathematics*, by Dr. W. S. Snyder, University of Tennessee.

The theory of the Burkill integral can be utilized to give a unified treatment of Rieman integration in one or more dimensions, of line or surface integrals, of arc length, continuous functions of bounded variation, and the Jordan theory of measure. The following two theorems are the essential tools which lead to all the results usually studied under the above headings in a course in Advanced Calculus. Theorem I: If f is a continuous function of intervals on the line and increases

by subdivision then its Burkill integral exists but may have the value $+\infty$. Theorem II: If f is a function of intervals (n-dimensional) which increases by subdivision and if a constant K exists such that for any interval I, $|f(I)| \leqq K|I|$, then the Burkill integral of f exists.

For precise definitions of the concepts involved, see Saks, *Theory of the Integral*, 2nd Ed., Chap. V, §3.

5. *The old and new methods in the theory of elasticity*, by Professor Tadeusz Leser, University of Kentucky.

After introducing the basic concepts and assumptions of the theory of elasticity, the author presented the tensor notation as compared with the scalar notation and also discussed the increasing use of conformal mapping for problems of twisting and bending.

6. *Inversion with respect to a cubic of the syzygetic pencil*, by Miss Elsie T. Church, University of Kentucky.

With base curve as a cubic of the pencil of syzygetic cubics,

$$F = x_1^3 + x_2^3 + x_3^3 - 3\lambda x_1 x_2 x_3 = 0,$$

an inversion analogous to quadric inversion was defined and the equations of the corresponding cubic transformation were derived. These equations are $x_1 : x_2 : x_3 = f_1 : f_2 : f_3$, where

$$f_1 = \lambda x_1'^2 x_2' - x_1' x_3'^2, \qquad f_2 = \lambda x_1' x_2'^2 - x_2' x_3'^2, \qquad f_3 = x_1'^3 + x_2'^3 - 2\lambda x_1' x_2' x_3'.$$

The curves $f_1 = 0$, $f_2 = 0$, $f_3 = 0$, intersect in seven common points and thus the transformation sets up a (1, 2) correspondence. Attention was called to the interesting results obtained with these seven points when λ is assigned the values 1, ω, ω^2, respectively.

7. *The knight's move on an infinite chess board*, by Professor A. W. Goodman, University of Kentucky.

A sequence of moves of the knight is called a grand tour if the knight occupies each square once and only once. Kurschak has proved that a grand tour is possible on an infinite chess board, *Acta Szeged*, vol. 4, pp. 12–13. Let $[r, s]$ denote the move of a generalized knight, where $[2, 1]$ is the ordinary knight move. A necessary condition for a grand tour is obviously $(r, s) = 1$ and $r + s$ odd. It is not known whether these conditions are sufficient.

AUGHTUM S. HOWARD, *Secretary*

THE APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The tenth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Manhattan College in New York City, New York, on Saturday, April 7, 1951. Dr. I. A. Dodes, the High School Vice-Chairman, presided at the morning session; Brother Bernard Alfred, Chairman of the Section, presided at the business meeting which followed the morning session, and he and Professor James Singer, Collegiate Vice-Chairman, presided at the afternoon session.

One hundred and twelve persons attended the meeting, including the following sixty-four members of the Association:

Brother Bernard Alfred, A. A. Bennett, Samuel Borofsky, C. B. Boyer, A. B. Brown, J. H. Bushey, Jewell H. Bushey, A. J. Carlan, Charles Clos, L. W. Cohen, T. F. Cope, W. H. H. Cowles, P. H. Daus, I. A. Dodes, J. N. Eastham, W. H. Fagerstrom, William Feller, Edward Fleisher, R. M. Foster, Leona Freeman, Harriet Griffin, George Grossman, G. C. Helme, W. M. Hirsch,

J. H. Hlavaty, T. R. Humphreys, L. C. Hutchinson, H. S. Kieval, J. J. Kinsella, R. J. Kohlmeyer, David Kotler, H. C. Kranzer, Helen Kutman, Martin Maltenfort, D. May H. Maria, Audrey Michaels, Martin Milgram, F. H. Miller, A. J. Mortola, W. R. Murray, D. S. Nathan, M. A. Nordgaard, C. J. Oberist, Eugene Odin, Walter Prenowitz, Hymen Rensin, Moses Richardson, John Riordan, Rose Roll, H. D. Ruderman, J. P. Russell, Charles Salkind, Arthur Schach, Abraham Schwartz, Aaron Shapiro, E. I. Shapiro, L. G. Sigler, James Singer, C. G. Solky, Mildred M. Sullivan, R. L. Swain, P. M. Treuenfels, Alan Wayne, M. E. White.

The following officers were elected for the coming year: Chairman, James Singer, Brooklyn College; Collegiate Vice-Chairman, L. F. Ollmann, Hofstra College; High School Vice-Chairman, E. I. Shapiro, Abraham Lincoln High School; Secretary, H. S. Kieval, Brooklyn College; Treasurer, Aaron Shapiro, Midwood High School. A resolution was passed which provided that the by-laws of the Section be interpreted to include Junior and Community Colleges in the naming of a representative from each collegiate department of mathematics to the Executive Committee of the Section. A resolution, that a second meeting of the Section be held in the fall of each year to be devoted to short papers, was referred to the Executive Committee for further study. The eleventh annual meeting will be held in the Spring of 1952.

Brother Bonaventure Thomas, President of Manhattan College, welcomed the people at the meeting, and then the following papers were presented.

1. *A theory of distribution*, by Professor Claude Chevalley, Columbia University, introduced by the Secretary.

The main object of Schwarz's theory of distributions is to put on a rigorous and uniform mathematical footing certain types of computational methods which are in frequent use by the physicists, like, for instance, the Heaviside Calculus.

One of the most frequent sources of difficulty in trying to make simple and general statements in analysis is the fact that a given function, even if it is continuous, does not necessarily have a derivative. The distributions are a class of mathematical objects wider than the class of functions, and inside which the operation of derivation is possible without any restriction. Thus, in particular, a continuous function always has derivatives of all orders; these derivatives are of course not functions in general, but distributions.

In order to define the notion of distribution, one introduces first the set D of indefinitely differentiable functions which are zero outside bounded sets (the bounded set depending on the function). This set is a vector space: the sum of two functions in D is in D, and the product by a constant of a function in D is in D. The distributions are then defined to be the linear functionals on D which are continuous relatively to a suitable notion of convergence for functions in D. A function $f(x)$ is identified with the functional which assigns to every g in D the number $\int_{-\infty}^{\infty} f(x)g(x)dx$. The derivative of a distribution S assigns to every function g in D the number $-S(dg/dx)$.

2. *Changing philosophy and content in tenth year mathematics*, by Dr. J. H. Hlavaty, Bronx High School of Science.

In the prolonged attempt of American secondary schools to adapt the curriculum to fit the ever-broadening educational aims, the continuous transformations in the philosophy and content of instruction in plane geometry has been a particularly instructive sidelight. A study of the recommendations of authoritative national bodies in the field of general education and in the field of mathematical instruction and a study of recent representative texts in plane geometry indicate

the following major attempts to reform instruction in demonstrative geometry: (1) Improving the methods of instruction; (2) Reformulating the aims of instruction; (3) Teaching geometry with a stress on practical applications; (4) Teaching geometry to develop the ability to think clearly; (5) The integrated mathematics movement; (6) The arithmetization of geometry; (7) Teaching geometry as an example of an empirical science.

However, it has proved impossible to rationalize the instruction in traditional demonstrative geometry either from the point of view of general education or from that of mathematical instruction. This failure is traceable to the fact that the reformers have in all cases retained traditional geometry or its method as the core of the mathematical instruction of the tenth year. It is suggested that there is a need for a course in general mathematics through the tenth year. In the tenth year the course should stress: (1) Methods of thinking (induction, deduction, statistical inference, "clear thinking"); (2) The uses of mathematics as a means of extending our understanding of the world (elementary statistics, trigonometry, spherical geometry); (3) The beginnings of these methods that lead to analytic geometry and the calculus.

3. *Changes in secondary school mathematics from 1900 to 1950*, by Mr. Joseph Orleans, George Washington High School, introduced by the Secretary.

At the beginning of the twentieth century the syllabi in mathematics in the American High Schools were determined largely by the requirements for entering the colleges. In 1902, a committee of the American Mathematical Society made recommendations for the content of the elementary algebra, the higher algebra, plane and solid geometry, and trigonometry. This report was described by David Eugene Smith as 'rather inclusive through its lack of precision.' It was to serve fairly well as starting point for reform.

In 1900 the College Entrance Examinations Board had come into being and gradually began through its examinations to exert its influence on syllabi.

In 1908 the work of the International Commission on the Teaching of Mathematics brought to the attention of teachers that other countries were ahead of us in the modernization of syllabi. In 1916, the National Committee on Mathematical Requirements began (the report appeared in 1923) with suggestions for the elimination of non-essentials, the retention of those things which should best meet the needs of pupils, and the introduction of such modern material as would strengthen the work without making it unreasonably difficult. In 1922, the College Entrance Examinations Board issued revised requirements which represented the combined judgment of the colleges and the secondary schools, and which set a new standard to eliminate much of the work for which no reasonable justification could be found. In the meantime the junior high school was developing with the introduction of informal geometry and the beginning of algebra earlier than the ninth year.

In 1930, one heard again the complaint that courses in secondary mathematics were still far from satisfactory. Recommendations were made for curriculum construction based on difficulties which pupils experience in the study of mathematics, and thus to eliminate the difficulties: (1) the introduction of arithmetic into the high school, (2) the teaching of algebra of a practical character in earlier years, (3) the stress on intuitional geometry.

Then came World War II with a wild scramble everywhere to stress the importance of mathematics for everybody with special courses—war courses—and refresher work in arithmetic, in basic mathematics, and related mathematics. In 1945, there was published the Report of the Commission on Post War Plans with the recommendations that the high school must come to grips with its dual responsibility, (1) to provide sound mathematical training for our future leaders in science, mathematics, and other learned fields, and (2) to insure mathematical competence for the ordinary affairs of life for all citizens as a part of a general education.

The questions that arise are, (1) What is the place of the able students in the present situation?, (2) What do schools do and what can they do to provide for these students?, (3) What is the effect upon their development and their preparation for work in higher institutions of learning?

4. *On chance fluctuations*, by Professor William Feller, Princeton University.

The speaker discussed fluctuations in waiting lines, queues, etc. He emphasized that the fluctuations are much larger than generally expected, and that they easily produce spurious effects resembling trends.

5. *Physical concepts in freshman mathematics*, by Professor A. A. Bennett, Brown University.

Texts and teachers currently use many terms dependent upon the human body, the solar system or clocks, in defining at freshman level, supposedly mathematical notions. On the other hand they often insist that only numbers (with their relations and operations) enter into equations. This view rejects the possible role of vectors in equations, and accepts scalars only when they are dimensionless or have been referred to arbitrarily specified units. It ignores all such rigorous studies as abstract group theory, topology, etc. But other views are possible. One may deal with "denominate numbers" either with or without reference to prior chosen units. There seems often great gain in so doing.

H. S. KIEVAL, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-third Summer Meeting, Michigan State College, East Lansing, Michigan, September 1–2, 1952.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Waynesburg College, Waynesburg, Pennsylvania, May 10, 1952.

ILLINOIS, Western Illinois State College, Macomb, May 9–10, 1952.

INDIANA, Indiana University, Bloomington, May 3, 1952.

IOWA, Coe College, Cedar Rapids, April 18–19, 1952.

KANSAS, Bethany College, Lindsborg, March 29, 1952.

KENTUCKY, University of Kentucky, Lexington, April 19, 1952.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Virginia Military Institute, Lexington, April 26, 1952.

METROPOLITAN NEW YORK, Hofstra College, Hempstead, New York, April, 1952.

MICHIGAN, University of Michigan, Ann Arbor, April 12, 1952.

MINNESOTA, College of St. Catherine, St. Paul, May 10, 1952.

MISSOURI, Lindenwood College, St. Charles, May 2, 1952.

NEBRASKA, University of Nebraska, Lincoln, May 3, 1952.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 19, 1952.

OKLAHOMA

PACIFIC NORTHWEST, University of Oregon, Eugene, June 20, 1952.

PHILADELPHIA

ROCKY MOUNTAIN, Western State College, Gunnison, Colorado, May 23–24, 1952.

SOUTHEASTERN, Georgia Institute of Technology and Agnes Scott College, Atlanta, March 21–22, 1952.

SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 8, 1952.

SOUTHWESTERN, University of Arizona, Tucson, April 11–12, 1952.

TEXAS, East Texas State Teachers College, Commerce, April 25–26, 1952.

UPPER NEW YORK STATE, Hobart and William Smith Colleges, Geneva, May 10, 1952.

WISCONSIN, Milwaukee, May 10, 1952.

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